2010 Qualifying Exams

Department of Physics and Astronomy, UNC Chapel Hill

Quantum Mechanics, part I: May 10, 9:00am-12:00pm

Choose 3 out of 5 problems

Start a new page for each problem. Label each page with the subject problem number and your PID, but not your name.

QMI-1 Three Hermitian operators satisfy the following commutations relations: $[\hat{A}, \hat{C}] = [\hat{B}, \hat{C}] = 0$ and $[\hat{A}, \hat{B}] \neq 0$.
Show that the spectrum of the operator $\hat{C}$ contains degenerate eigenvalues.

QMI-2 Here is a guided proof that there are wave functions that oscillate back and forth in a one-dimensional harmonic oscillator without spreading:

(a) Consider a state of the form

$$|\psi\rangle = e^{-i\hat{p}\,\lambda/\hbar}\,|\phi\rangle\ ,$$

where $\hat{p}$ is the usual momentum operator, at time $t_0 = 0$. How is this state related to $|\phi\rangle$?

(b) Show that in the Schrödinger picture, the state vector $|\psi(t)\rangle$ for $t > 0$ is given by

$$|\psi(t)\rangle = e^{-i\hat{p}(-t)\,\lambda/\hbar}\,|\phi(t)\rangle\ ,$$

where $|\phi(t)\rangle$ is the Schrödinger-picture state that evolves from $|\phi\rangle$, and $\hat{p}(t)$ is the *Heisenberg-picture* operator that evolves from the usual momentum operator $\hat{p}$.

(c) Now let the system be a harmonic oscillator with oscillator frequency ω. Find $\hat{p}(-t)$ in terms of the $\hat{p}$ and the usual position operator $\hat{x}$. (*Hint:* For linear systems like the oscillator, the solutions to the Heisenberg equation of motion are the same as the classical solutions.)

(d) Now suppose $|\phi\rangle$ is the oscillator ground state. Use the fact that

$$e^{A+B} = e^A e^B e^{-1/2[A,B]} \qquad (1)$$

(if $[A, B]$ commutes with both A and B) and the form of the ground state wave function

$$\langle x|\phi\rangle \propto e^{-\frac{m\omega}{2\hbar}x^2} \qquad (2)$$

to show that while the wave function $\langle x|\psi(t)\rangle$ moves, it doesn't spread.

QMI-3 A particle of mass m is subject to the one-dimensional potential $U(x) = -\alpha\delta(x-a)$ for $x > 0$, and $U(x) = \infty$ for $x < 0$.

(a) Find the number of bound states as a function of the parameter $\alpha ma/\hbar^2$.

(b) Is the effective force acting between the particle and the wall repulsive or attractive?

QMI-4 Arbitrary spin operator. A non-interacting, spin-1/2 particle has an angular momentum component that is determined to be pointing in the $+\hat{z}$ direction. The spin is then measured along an arbitrary direction, $\hat{n}$.

(a) What is the total intrinsic angular momentum of this particle?

(b) Find the expectation value for the angular momentum measurement in the $\hat{n}$-direction.

(c) Assume for this and the subsequent question that the angle between the $+\hat{z}$ vector and $\hat{n}$ is $\pi/4$ radians. Find the probability that the angular momentum component measured in $\hat{n}$-direction is positive.

(d) A measurement of the angular momentum component pointing in the $\hat{n}$ direction is performed, and it is found to be greater than zero. What are the possible outcomes for another subsequent measurement of the angular momentum component along the z-axis and what are their probabilities?

Reminder:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

QMI-5 Non-rigid rotator. Consider a diatomic molecule consisting of identical atoms, each of mass m, and with a separation between nuclei of r_0 when the molecule is in the zero angular momentum state ($L = 0$). Assume that the binding force between the atoms can be modeled as a spring with spring constant k. Answer the following questions:

(a) The rotational excitation levels in most diatomic molecules have much lower energies than that of the vibrational excitation levels. Find a relationship between m, r_0, and k that has to be satisfied for this to be true.

(b) Assume that the molecule satisfies the condition that you found in part (a). It is placed in a gas with enough thermal energy to excite the rotational but not vibrational levels. By correcting for the fact that this molecule is *not* a rigid rotator, find the approximate frequency of a photon emitted during a transition from the first excited to ground rotational states.

2010 Qualifying Exams
Department of Physics and Astronomy, UNC Chapel Hill

Statistical Mechanics: May 10, 9:00am-12:00pm

Choose 3 out of 5 problems

Start a new page for each problem. Label each page with the subject problem number and your PID, but not your name.

SM-1 Consider the ideal Fermi gas where the single-particle eigenstates and eigenvalues of the Hamiltonian are s and ε_s, respectively.

(a) Based on the quantum properties of fermions, derive the grand partition function of such system with given fugacity z, volume V, and temperature T is given by

$$\mathcal{Z}(z,V,T)=\prod_s\left(1+ze^{-\varepsilon_s/k_BT}\right)$$

(b) Write down the relationship between $\mathcal{Z}$ and the grand canonical potential $\Phi=U-TS-\mu N=-PV$.

SM-2 Consider the internal rotational degree of freedom of a diatomic molecule with a moment of inertia I. Its Hamiltonian is given by $\hat{H}_{rot}=\hat{L}^2/(2I)$ and $\hat{L}^2|l,m\rangle=l(l+1)\hbar^2|l,m\rangle$ with $l=0,1,2,\ldots,\ \ m=-l,-l+1,\ldots,l$.

(a) Write down the expression of its partition function associated with rotation (ignore the nuclear spin effect).

(b) Based on the result of (a), calculate $U_{rot}=U_{rot}(T)$ at the low temperature limit.

(c) Now consider molecular H_2 where the effect of nuclear spins has to be taken into account. Protons are spin-1/2 fermions and the two spins can form singlet and triplet states. Write down the partition function of rotation with the effect of the nuclear spin states included.

SM-3 Calculate the Joule-Thomson coefficient $(\partial U/\partial V)_T$ where U is the internal energy for a non-ideal gas described by the van der Waals' equation of state $P=RT/(V-B)-a/V^2$.

SM-4 The average energy of a system in thermodynamic equilibrium is $\langle E \rangle$.

(a) Show that the mean square of the energy deviation from its average value equals $\left\langle \left(E-\langle E\rangle\right)^2\right\rangle = k_B T^2 C_V$.

(b) Estimate, for a system of N>>1 particles, the relative deviation $\left\langle \left(E-\langle E\rangle\right)^2\right\rangle / \langle E^2 \rangle$ in the high-temperature limit.

SM-5 There have been recent attempts to interpret gravity as an entropic force (and possibly there are gaps in physical reasoning. But let us play along).

(a) Consider the entropic force $F\delta x = T\delta S$, where δx denotes an infinitesimal spatial separation, and fix δx by the Compton wavelength $\delta x = \hbar/(mc)$ for the particle of given mass m, for the infinitesimal increase of entropy $\delta S = 2\pi k_B$. Let us postulate the entropic change to be linear with the change in distance $\delta S = 2\pi k_B (mc/\hbar)\delta x$. Now, let us further adopt the famous formula $k_B T = \hbar a/(2\pi c)$ for the (Unruh) temperature associated with a uniformly accelerated (Rindler) observer with acceleration a. Derive the second law of Newton $F=ma$.

(b) Consider a point particle of mass M at the origin. One can associate an energy $E = Mc^2$ with this mass. But, let us assume that this energy is equal to the energy of N degrees of freedom at temperature T on the surface of the sphere of radius r, where $N = Ac^3/(G_N \hbar)$ (with $A = 4\pi r^2$). Use the equipartition theorem, the Unruh temperature expression, and the result in (a) to derive Newton's law for the gravitational force: $F = G_N Mm/r^2$.

2010 Qualifying Exams

Department of Physics and Astronomy, UNC Chapel Hill

Electromagnetism I: May 7, 1:30pm-4:30pm

Choose 3 out of 5 problems

Start a new page for each problem. Label each page with the subject problem number and your PID, but not your name.

EMI-1 What is the potential in a rectangular region bounded by $0 < x < a$, and $0 < y < b$, given that the boundary conditions on the potential are that it vanishes on the two edges that have $y = constant$, that it is constant ϕ_0 (not zero) on the edge that has $x = a$, and that its derivative vanishes on the edge $x = 0$. Give a physical explanation for the mathematical content of your answer.

EMI-2 A thin spherical shell of radius R carries a uniform surface charge density σ. The shell is rotating along z axis with angular frequency ω.
(a) Write down the surface current density associated with the rotating charge, and the boundary condition for the magnetic field cross the shell.

(b) Show that the magnetic scalar potential is given by
$\Phi_m(r, \theta, \phi) = \frac{2}{3}\sigma\omega Rr cos\theta$ for $r < R$, and
$\Phi_m(r, \theta, \phi) = \frac{1}{3}\sigma\omega\frac{R^4}{r^2} cos\theta$ for $r > R$.

(c) Calculate the magnetic field both inside and outside R.

EMI-3 (a) Calculate the electric potential around a symmetric quadrupole comprising charge $-2q$ at the origin and charges $+q$ at distance a above and below on the z axis. Obtain the usual far field approximation for large r at the end.

(b) Now for same quadrupole inside a grounded conducting sphere of radius $b > a$, calculate the potential for all points $r > a$. Again obtain the far field approximation for all $r >> a$. (Hint: consider image charges.)

EMI-4 (a) Write the differential form of Maxwell's equations in vacuum with sources.

(b) When $B_i = \epsilon_{ijk}\partial^j A^k$ and $E_i = -\partial_i\phi + \partial_0 A_i$, show which two of Maxwell's equations are automatically satisfied.

(c) Show that $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2\vec{A}$.

Hint: Use $(\vec{\nabla} \times \vec{A})_i = \epsilon_{ijk}\partial^j A^k$ and $\epsilon_{ijk}\epsilon_{\ell mk} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}$.

(d) Give the gauge transformation on the vector potential A^k that leaves invariant the magnetic field B_i. Show that the gauge transformation $\phi \longrightarrow \phi - \frac{\partial \Lambda}{\partial t}$ combined with a gauge transformation on A^k leaves invariant the electric field E_i.

e) Write the remaining two Maxwell's equations, that are not automatically satisfied, in terms of the potentials A_i and ϕ in the gauge $\vec{\nabla} \cdot \vec{A} = 0$, when $A_i(\vec{x})$ and $\phi(\vec{x})$ are independent of time.

EMI-5 A thin uniform metal disk with mass density ρ is balanced on top of a much larger diameter conducting sphere of in a uniform gravitational field. The radius of the sphere is R and it never moves. Charge is slowly added to the sphere. At what total charge Q on the sphere would the disc starts to lift off from the sphere?

2010 Qualifying Exams

Department of Physics and Astronomy, UNC Chapel Hill

Classical Mechanics: May 7, 1:30pm-4:30pm

Choose 3 out of 5 problems

Start a new page for each problem. Label each page with the subject problem number and your PID, but not your name.

CM-1 Consider a mechanical system for which the potential energy $V(\vec{r_1}, \vec{r_2}...)$ is a homogeneous function of the coordinates $\vec{r_i}$ of degree n. Let us scale all the coordinates by a factor of α and the time by a factor of β.

(a) Show that for $\beta = \alpha^{1-\frac{n}{2}}$, the equation of motion is NOT changed under these scaling operations.

(b) Show that for $\beta = \alpha^{1-\frac{n}{2}}$, the same set of equations of motion permits a series of geometrically similar paths with the times of motion between corresponding points being given by the ratio $\frac{t'}{t} = \left(\frac{l'}{l}\right)^{1-\frac{n}{2}}$, where $\frac{l'}{l}$ is the ratio of linear dimensions of the two paths. What else is required besides scaling of the potential and time in order to permit self-similiar motion?

(c) Show that for harmonic oscillators, the period of oscillations is independent of their amplitudes (by using part b).

(d) Show that in free fall under gravity, the time of fall goes as the square root of the initial altitude (by using part b).

CM-2 If the mass and spring constant in a harmonic oscillator have a particular time dependence, one can arrive at the time-dependent Hamiltonian

$$H = f(t)\left[\frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2\right],$$

where ω is a constant and $f(t)$ is the derivative of some other well behaved function $g(t)$.

(a) Write the Hamilton Jacobi equation for Hamilton's principal function $S(q, \alpha, t)$, where α is the "new" momentum.

(b) Solve the equation to find $q(t)$ in terms of the usual constants α and β. How would you describe the physical meaning of the constant α?

CM-3 A particle of mass m moves in one dimension (along x) under a potential

$$V = a^2 x^4 - 2b^2 x^2,$$

where a and b are constant parameters.

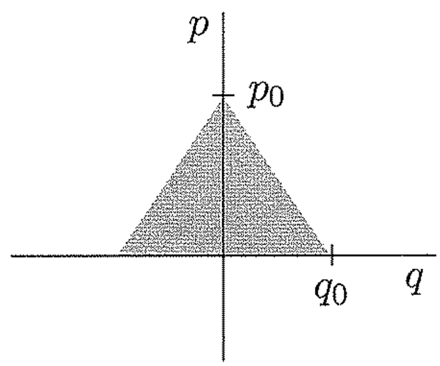
(a) Determine the locations of the equilibria.

(b) Find the frequency of small amplitude motion about stable equilibria.

(c) Find the exponential growth rate for small amplitude motion away from the unstable equilibrium.

(d) Derive the Hamiltonian and sketch the surfaces of constant energy in phase space.

CM-4 The Liouville theorem states that areas in phase space are conserved. Consider an ensemble of free particles and the initial $t = 0$ phase space distribution drawn below:



(a) Without using the Liouville theorem itself (unless you want to derive it), show that at time t the region has evolved into another region with the same area.

(b) Liouville's theorem can be proven by showing that the transformation from q_0, p_0 to $q(t), p(t)$ is canonical for any time t. Show explicitly that the transformation is canonical in this simple example.

CM-5 Consider a particle of mass m that is confined to the surface of a torus and is acted upon by a uniform gravitational acceleration g. Let the torus have minor radius b and major radius a. Positions on the torus are described by two angle coordinates, θ and ϕ. The angle ϕ is an azimuthal coordinate that circles the symmetry axis z (i.e., goes around the torus the long way) and θ goes through 2π as it circles the circular cross section of radius b. The transformation between cartesian coordinates and toroidal coordinates is

$$\begin{aligned} x &= (a + b\sin\theta)\cos\phi \\ y &= (a + b\sin\theta)\sin\phi \\ z &= b\cos\theta, \end{aligned}$$

and the line element (or metric) for the surface geometry of the torus is

$$ds^2 = b^2 d\theta^2 + (a + b\sin\theta)^2 d\phi^2,$$

which can be used to find the velocity (tangent vector) of any trajectory on the toroidal surface.

(a) Obtain the Lagrangian for motion on the toroidal surface.

(b) Determine the symmetries of the Lagrangian and the conserved quantities.

(c) Assuming there is some motion in the ϕ direction, obtain the effective potential for motion in θ.

(d) Assuming you are told that, under steady motion in ϕ at a certain rate Ω, the particle maintains a constant equilibrium angle θ_c. Given some θ_c, use the equations for equilibium to determine expressions for the values of the conserved quantities.

2010 Qualifying Exams

Department of Physics and Astronomy, UNC Chapel Hill

Quantum Mechanics, part II: May 10, 1:30pm-4:30pm

Choose 3 out of 5 problems

Start a new page for each problem. Label each page with the subject problem number and your PID, but not your name.

QMII-1 Atomic Resonance. Consider a single electron that experiences a static central potential $V(r)$. We add a weak, external, time-varying magnetic field with a corresponding vector potential $\mathbf{A}(r,t)$. Use a specific case of the Lorentz gauge for this problem:

$$\nabla \cdot \mathbf{A} = \frac{1}{c}\frac{\partial \phi}{\partial t} = 0$$

(a) Write down the Hamiltonian for this system.

(b) Assume that the perturbation due to $\mathbf{A}$ is much smaller than the energy scale imposed by V. Use this property and the Lorentz Gauge condition to write the Hamiltonian as a sum of a static, unperturbed part and a smaller, time-dependent part. You may find the following vector identity useful:

$$\nabla \cdot (\mathbf{A}f) = (\nabla \cdot \mathbf{A})f + \mathbf{A} \cdot (\nabla f)$$

(c) Assume that the weak, time-varying potential is of the form:

$$\mathbf{A}(t) = \mathbf{A_0} \cos \omega t$$

where $\mathbf{A_0}$ may be assumed constant on the scale of the electron's wavefunction. Let a solution to the unperturbed potential from part (b) be written as:

$$\psi(t) = \sum_k c_k(t)\psi_k(t)$$

where ψ_k are the eigenfunction of the unperturbed Hamiltonian. Assume that the system is in eigenstate m of the unperturbed Hamiltonian at $t = 0$. Find the probability that the system will be in eigenstate n at a later time, given that $m \neq n$. You may collect all the time-independent coefficients (constants, expectation values, etc.) into one constant, N, that you are not required to evaluate.

(d) Assume that the value of the driving frequency, ω, is very close to the value of

$$\omega_{mn} = (E_m - E_n)/\hbar$$

and simplify the expression you found in part (c) further.

(e) Under the assumption from part (d), sketch the transition probability for the transition from state m to state n as a function of ω_{mn}. Comment on and discuss your result.

QMII-2 Morse Potential. A phenomenological formula that describes the interaction potential between two atoms in a diatomic molecule is the so-called *Morse Potential*:

$$V(r) = D(1 - e^{-\alpha(r-r_0)})^2$$

where r is the separation between the atomic nuclei.

(a) Sketch this potential and provide a physical interpretation of the parameters D and r_0.

(b) Provide a *qualitative* sketch of the energy-levels of this potential.

(c) Find a potential to approximate the given Morse potential and find the first non-zero perturbation theory correction to the ground state for that potential.

You may find the following information useful:

$$\int_0^\infty x^{2n} e^{-px^2} dx = \frac{(2n-1)!!}{2(2p)^n}, \quad n!! = n \times (n-2) \times (n-4) \times \ldots$$

$$\int_0^\infty x^{2n+1} e^{-px^2} dx = \frac{n!}{2p^{n+1}}$$

$$\psi_n(x) = (\frac{m\omega}{\pi\hbar})^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\nu) e^{-\nu^2/2} \; ; \nu = \sqrt{\frac{m\omega}{\hbar}} x$$

QMII-3 A particle of mass m is temporarily captured in a state with angular momentum $l > 0$ and energy $E > 0$ inside a sphericallly-symmetrical well of depth V and radius R. Neglecting the centrifugal barrier within the well, evaluate the half-life time τ of such a metastable state.

QMII-4 Particles in a well. Three identical, spin-$\frac{1}{2}$ particles, each with mass m, are trapped in an isotropic three dimensional harmonic oscillator well with a classical angular oscillation frequency ω. The only interaction that the particles experience amongst themselves is the coupling between their intrinsic magnetic dipole moments ($\vec{\mu}$). The potential energy of this coupling is equal to the dot product of the two dipole moments, multiplied by a constant, β. It does not depend on the distance between the particles.

(a) Write down the hamiltonian for this system.

(b) Assume that the spatial component of the system's wavefunction corresponds to its lowest energy state allowed by symmetry principles. Find the energies and degeneracies of all the allowed states in this case.

QMII-5 Consider scattering of a plane wave $|k>$ off a potential with a characteristic length a. It is know that the phase shifts δ_l for all spherical partial waves are given by the expression

$$\sin \delta_l = \sqrt{\frac{(ka)^l}{(2l+1)l!}}$$

(a) Considering the p partial wave only, what is the ratio of differential scattering cross section in the forward direction to that of the backward direction?

(b) If the first resonance scattering is observed for the p-wave at certain energy, what would be the s-wave scattering cross section at this particular energy?

(c) What would be the total differential cross section if all partial waves are included? Calculate the total scattering cross section for $ka = 1$.

2010 Qualifying Exams

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Electromagnetic Theory II: May 10, 1:30pm-4:30pm

Choose 3 out of 5 problems

Start a new page for each problem. Label each page with the subject problem number and your PID, but not your name.

EMII-1. A particle of mass m and Lorentz factor γ scatters off a particle of equal mass that was initially stationary. The collision is elastic and the first particle scatters off at an angle θ relative to its initial direction of motion. Determine the Lorentz factor γ' of this particle as a function of $\cos\theta$ and γ.

EMII-2. An undulator is a device for producing coherent electromagnetic radiation from a beam of relativistic electrons (i.e., free electron laser). In an undulator, a relativistic electron passes through a region with alternating magnetic field direction. The alternating magnetic field causes the electron to wiggle in the transverse direction, and thereby radiate.

Assume the magnetic field in the device varies sinusoidally in the y-direction,

$$\vec{B} = (0, B_0 \cos(kz), 0),$$

with the magnet spacing related to k. Let the electron's velocity be primarily along the z-direction but perturbed by the magnetic field,

$$\vec{v} = (u(t), 0, v).$$

Here the (longitudinal) z-component is $v \simeq c$ and is unaffected by the magnetic field at first order.

(a) In whatever frame you choose, use the equations of motion to express the time dependence of the x and t components of the four velocity.

(b) What is the solution for the time dependence of the Lorentz factor?

(c) In whatever frame you choose for the calculation, compute the total average radiated power from a single relativistic electron as seen in the lab frame of the undulator.

EMII-3. A general expression for the spectral and angular distribution of energy radiated from a relativistic electron is

$$\frac{dW}{d\omega d\Omega} = \frac{e^2\omega^2}{4\pi^2 c}\left|\int dt'\vec{n}\times(\vec{n}\times\vec{\beta})\exp[i\omega(t' - \vec{n}\cdot\vec{x}(t')/c)]\right|^2,$$

which is derived using the Lienard-Wiechart expression for the radiative part of the electric field.

A neutral particle like the Z^0 can decay into an electron-positron pair. If the Z^0 is at rest when it decays, the pair of particles fly in opposite directions but with equal speeds β.

Let the electron have velocity

$$\vec{\beta}_{e^-} = (0,0,\beta) \qquad \text{for} \quad t' > 0,$$

and thus have position vector

$$\vec{x}_{e^-}(t') = (0,0,c\beta t') \qquad \text{for} \quad t' > 0.$$

The positron's velocity and position vector are similar but with the sign of β reversed. Let the observation direction be taken to be $\vec{n} = (\sin\theta, 0, \cos\theta)$.

(a) Calculate the appearance radiation for pair production.

(b) Consider the nonrelativistic limit of this result. What does the angular dependence and (lowest) power of velocity in this limiting expression suggest?

EMII-4. Recalling Faraday's and Ampere's laws for a medium with non-trivial permittivity,

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}, \tag{1}$$

$$\vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial\vec{D}}{\partial t}, \tag{2}$$

assume the presence of a transverse plane electromagnetic wave $\vec{E} = \vec{E}_0\exp(i\vec{k}\cdot\vec{x} - i\omega t)$. Let the permittivity be scalar and given by

$$\epsilon = 1 - \frac{\omega_p^2}{\omega(\omega + i\nu)}. \tag{3}$$

(a) Derive the dispersion relation.

(b) Using the dispersion relation and assuming a real driving frequency ω with $\omega > \omega_p$ and $\omega \gg \nu$, calculate the damping distance (or e-folding distance) for a plane wave propagating in this medium.

EMII-5. An electron of charge e and mass m moves in a circular orbit under the Coulomb force produced by a proton. The average potential enenergy $< V(r) >$ is related to the total energy by $E =< V > /2$. Suppose, as it radiates, the electron continues to move on a circle.

(a) Show that the power radiated is given by $-\frac{dE}{dt} = \frac{2e^2}{3c^3}\left(\frac{e^2}{mr^2}\right)^2$.

(b) Show that it takes the electron $t = \frac{m^2c^3r_{in}^3}{4e^4}$ to hit the proton if it starts from an initial radius of r_{in}. Assume you have never heard of quantum mechanics.

2010 Qualifying Exams

Department of Physics and Astronomy, UNC Chapel Hill

Astrophysics I: May 10, 1:30pm-4:30pm

Choose 3 out of 5 problems

Start a new page for each problem. Label each page with the subject problems number and your PID, but not your name.

1. Determine the fraction of hydrogen atoms that are ionized at the center of the sun, assuming ionization equilibrium, $T = 15.8$ million K, and $n_e = 6.4 \times 10^{31}$ m^{-3}. Does your result agree with the fact that practically all of the sun's hydrogen is ionized at the sun's center? What are reasons for any discrepancy? Suppose the star has twice the metal content of the sun. Would the level of ionization in its center be higher or lower than in the center of the sun? Why?

2. Approximate a white dwarf of mass M and radius R as a degenerate core surrounded by an ideal gas atmosphere.

   (a) Show that the atmospheric pressure as function of temperature is in general

$$P(T) = \left(\frac{64\pi ac\mathcal{R}G}{51\kappa_0\mu}\right)^{1/2} (\Upsilon)^{1/2} T^{14/4}$$

   for mass/luminosity ratio Υ.

   (b) From this expression, show that the temperature profile throughout this atmosphere $r < R$ is

$$T(r) = \frac{4}{17}\frac{\mu}{\mathcal{R}} GM(\frac{1}{r} - \frac{1}{R})$$

3. The figure below show a schematic of a (theorist's) H-R diagram left blank except for the sun.

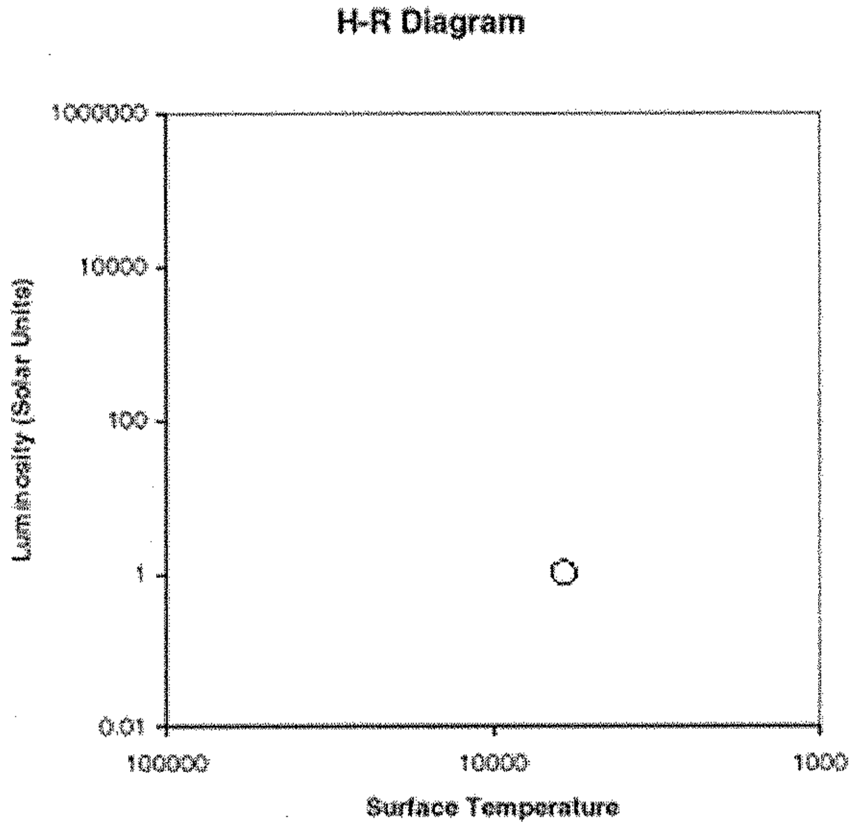

Figure 1: H-R diagram

(a) Sketch the main sequence in the diagram.

(b) Explain the procedure for comparing calculated luminosities and temperatures of stellar models to the observable quantities (color-magnitude).

(c) Assuming the mass-luminosity relationship on the main sequence is $L \propto M^{3.5}$, derive a relationship for lifetime on the main sequence under the assumption that all stars have the same fraction of their H available for nuclear burning. Then draw a + at the location of a star with ~1/700 the lifetime of the sun.

(d) Draw a * at the location of a star that has 100 times smaller luminosity than the sun and 10 times smaller radius.

(e) Draw an arrow showing the approximate direction the sun would move in the diagram if it cooled without changing its radius.

4. Consider a planetary transit across the disk of another star.

   (a) Using geometry and a relationship for limb darkening, plot the shape of the light curve.

   (b) What is the transit duration of a Jupiter analog orbiting at 0.3 AU across the center of the disk of a G2 main-sequence star? (Jupiter is 10× the diameter of the Earth.)

   (c) What is the maximum eclipse depth in % visible wavelength light attenuation for b)?

   (d) Assuming that the planet has temperature 900 K, what is the ratio of IR flux at 3 micron wavelength of the planet to star in b)?

5. Estimate the duration of the helium burning phase of the sun (i.e. when it is on the horizontal branch). Assume:

   (a) Its luminosity on this branch will be 100 times higher than it is now.

   (b) Each reaction fusing 3 helium nuclei into 1 carbon produces 1.8×10^{-12} J, which is 40% the energy produced in the fusion of 4 hydrogen to 1 helium.

   (c) The sun will start helium burning with 10 % of a solar mass of helium, and will fuse essentially all of it. Remember that each helium nucleus is about four times as massive as a hydrogen nucleus (helium mass = 6.7×10^{-27} kg).

   (d) Using these assumptions, how long will the Sun burn helium? You may give your answer in years or in relation to the main sequence lifetime of the sun, but please indicate which you mean.

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Department of Physics and Astronomy, UNC Chapel Hill

Astrophysics II: May 10, 1:30pm-4:30pm

Choose 3 out of 5 problems

Start a new page for each problem. Label each page with the subject problems number and your PID, but not your name.

ASTROII-1. Equation of State of a Degenerate, Ideal Fermi Gas

Consider a completely degenerate, ideal electron gas.

a. Write down an expression for the electron number density n_e in terms of the distribution function in phase space. Solve it, yielding n_e as a function of $x = p_F/m_e c$.

b. Assume that the mass density ρ is dominated by non-degenerate ions. Write down an expression for ρ as a function of x.

c. Write down an expression for the electron pressure P_e in terms of the distribution function. Solve it, yielding P_e as a function of x.

d. Series expand P_e in the relativistic limit. Keep only the leading term.

e. What then is the equation of state in the relativistic limit?

$\int_0^x \frac{x^4 dx}{(1+x)^{1/2}} = \frac{3}{8}\{x(1+x^2)^{1/2}(2x^2/3-1) + \ln[x+(1+x^2)^{1/2}]\}$

$\int_0^x (1+x)^{1/2} x^2 dx = \frac{1}{8}\{x(1+x^2)^{1/2}(1+2x^2) - \ln[x+(1+x^2)^{1/2}]\}$

ASTROII-2. Chandrasekhar Limit

Consider a white dwarf of radius R consisting of N fermions.

a. Write down an approximate expression for the typical distance q between fermions as a function of N and R. Ignore factors of order unity.

b. Using this and the uncertainty principle, write down an approximate expression for the typical momentum p of a fermion as a function of N and R. Ignore factors of order unity.

c. Using this, write down an approximate expression for the typical kinetic energy E_{KE} of a fermion in the relativistic limit as a function of N and R.

d. Write down an approximate expression for the typical gravitational potential energy E_{PE} of a fermion as a function of N and R. Keep in mind that although the pressure is dominated by electrons, the mass is dominated by baryons. Ignore factors of order unity.

e. Write down an approximate expression for the typical total energy E of a fermion as a function of N and R.

f. If N is small, $E > 0$ and can be minimized by increasing R until the fermions become non-relativistic. If N is large, $E < 0$ and can be minimized by decreasing R (i.e., the white dwarf collapses). Consequently, determine an approximate expression for the largest value of N that a white dwarf can have without collapsing. Evaluate it.

$\hbar = 1.1 \times 10^{-27}$ erg s

$c = 3.0 \times 10^{10}$ cm s^{-1}

$G = 6.7 \times 10^{-8}$ cm^3 g^{-1} s^{-2}

$m_B = 1.7 \times 10^{-24}$ g

ASTROII-3. White Dwarf Cooling

Consider a carbon white dwarf of mass M and of interior temperature T that is in excess of the crystallization temperature.

a. Write down an expression for the thermal energy per ion as a function of T.

b. Write down an expression for the total thermal energy of the white dwarf as a function of T and M.

c. Using this, write down an expression for the luminosity L of the white dwarf as a function of T and M.

d. Photon diffusion from the interior to the surface implies that:

$$L = (2 \times 10^{6}\,\mathrm{erg/s}) \left(\frac{M}{M_{\odot}}\right) T^{7/2}. \tag{1}$$

Using this, write down a differential equation for T as a function of time.

e. Solve it assuming that the initial temperature is much greater than T. Write down an expression for the age τ of the white dwarf as a function of T.

f. What is the interior temperature of a 0.65-$M_{\odot}$ white dwarf of luminosity 10^{31} erg? What is its age in years?

$k = 1.4 \times 10^{-16}$ erg K^{-1}

$m_u = 1.7 \times 10^{-24}$ g

$M_{\odot} = 2.0 \times 10^{33}$ g

ASTROII-4. Photodissociation

Consider the photodissociation of ^{56}Fe before a Type II supernova:

a. Each ^{56}Fe nucleus dissociates into 13 alpha particles and 4 neutrons. Write down an expression relating their chemical potentials.

b. For a Maxwell-Boltzmann gas:

$$n_i = g_i \left(\frac{m_i kT}{2\pi \hbar^2} \right)^{3/2} \exp \left(\frac{\mu_i - m_i c^2}{kT} \right). \tag{2}$$

where $g_{Fe} \approx 1.4$, $g_\alpha = 1$, and $g_n = 2$. Write down the Saha equation, where:

$$Q = (13 m_\alpha + 4 m_n - m_{Fe}) c^2 = 124.4\,\mathrm{MeV}. \tag{3}$$

c. Assuming that ^{56}Fe is the most abundant heavy nucleus, write down an expression relating n_α and n_n.

d. Using this, write down an expression relating the mass density ρ and the temperature T when half of the mass has been dissociated.

e. What is the mass density at which this occurs if $kT = 1$ MeV ($= 1.6 \times 10^{-6}$ erg)?

$m_u = 1.7 \times 10^{-24}$ g

$\hbar = 1.1 \times 10^{-27}$ erg s

ASTROII-5. Relativistic Beaming

Consider the jet of a very low-redshift ($z \ll 1$) gamma-ray burst. Assume that its bulk Lorentz factor as a function of observer-frame time is given by:

$$\Gamma = 100 \left(\frac{t}{1\,\mathrm{min}} \right)^{-3/8} \tag{4}$$

a. Assume that the jet is 0.2 radians across, that it is not expanding laterally with time, and that its center is pointed directly at us. At what observer-frame time, in hours, will the jet begin to fade in brightness more quickly?

b. At what observer-frame time would this occur if the same gamma-ray burst were at redshift 6.3?

c. Again assume that $z \ll 1$, but that the jet is not pointed directly at us. Assume that its center is pointed 0.2 radians away from us. At what observer-frame time, in hours, will the jet begin to brighten?

d. Since this is much longer than the jet's gamma-ray emitting phase, such events are likely missed by gamma-ray spacecraft, but might be picked up as "orphan" afterglows in optical surveys.

Suppose that in a 1-minute exposure you could detect regular afterglows typically for 1 hour, but orphan afterglows typically only for 15 minutes, since they take a while to brighten. Also assume that to this detection limit, one orphan afterglow appears (somewhere) in the sky every day. If your field of view is a large 1 square degree and you take 1-minute exposures for 10 hours each night, how many nights will it take you to detect an orphan afterglow?