Past qual problems

2015 SM1 $\mathcal{E} = \frac{pL}{h}$, $\frac{d\mathcal{E}}{dE} = \frac{d\mathcal{E}}{dp}\frac{dp}{dE} = \frac{L}{h}\frac{m}{p} = g(E)$

$$Q_1 = \int \frac{L}{h}\frac{m}{p} e^{-\beta\left(\frac{p^2}{2m} - \epsilon_0\right)} \frac{dE}{dp} dp = \frac{L}{h} e^{\beta\epsilon_0} \int_{-\infty}^{\infty} e^{-\beta/2m\, p^2} dp \quad , \quad x = \sqrt{\frac{\beta}{2m}}\, p = \frac{p}{\sqrt{2mkT}}$$

$$= \frac{L}{h}\sqrt{2mkT}\, e^{\beta\epsilon_0} \underbrace{\int_{-\infty}^{\infty} e^{-x^2} dx}_{\sqrt{\pi}} = \frac{L}{\lambda} e^{\epsilon_0/kT} \quad , \quad \lambda = \frac{h}{\sqrt{2\pi mkT}}$$

$$Q_N = \frac{1}{N!} Q_1^N = \frac{1}{N!}\left(\frac{L}{\lambda}\right)^N e^{N\epsilon_0/kT} \quad , \quad A = -kT\log Q_N = -kTN\left(\log\frac{L}{\lambda N} + \frac{\epsilon_0}{kT} + 1\right)$$

$\mu N = G = A + PV$. $P = -\frac{\partial A}{\partial L} = kTN/L$, $PV = PL = NkT$

$dU = TdS - PdV + \mu dN$
$dA = -SdT - PdV + \mu dN$
$dG = -SdT + Vdp + \mu dN$

$$\Rightarrow \mu = \frac{A + PL}{N} = -kT\log\left(\frac{L}{\lambda N}\right) - \epsilon_0 = kT\log\left(\frac{\lambda N}{L}\right) - \epsilon_0$$

SM2 $p_F = \left(\frac{3N}{8\pi V}\right)^{1/3} h$, $V = \frac{4}{3}\pi R^3 \Rightarrow p_F = \left(\frac{N}{2R^3}\right)^{1/3} h = \left(\frac{N}{2}\right)^{1/3}\frac{h}{R}$,

$$P = P_G = \frac{-\partial U_G}{\partial V}, \quad U_G = G\int_0^R \frac{\frac{4}{3}\pi r^3 \rho\, 4\pi r^2 dr\, \rho}{r} = G\left(\frac{M}{4/3\pi R^3}\right)^2 3\left(\frac{4}{3}\pi\right)^2 \int_0^R r^4 dr = \frac{3}{5}\frac{GM^2}{R}$$

$$U_G = \frac{3}{5}\frac{GM}{\left(\frac{3V}{4\pi}\right)^{1/3}} \Rightarrow P_G = \frac{1}{5}\frac{GM}{\left(\frac{3V^4}{4\pi}\right)^{1/3}}$$

$$T \to 0 \Rightarrow P = 2\frac{4\pi}{3h^3}\int_0^{p_F} p^4 dp \underbrace{\left[1 + \left(\frac{p}{m_e c}\right)^2\right]^{-1/2}}_{\frac{1}{\cosh\theta_f}} \frac{1}{m_e} \quad , \quad p_f = m_e c \sinh\theta_f$$

$$\epsilon = m_e c^2(\cosh\theta - 1) \qquad dp = m_e c\cosh\theta$$

$$P = \frac{8\pi}{3h^3}\int_0^{\theta_f} m_e^4 c^5 \sinh^4\theta\, d\theta = \frac{1}{5}\frac{GM^2}{R^4 \frac{4}{3}\pi}$$

$$\frac{\pi m_e^4 c^5}{3h^3} 8\int_0^{\theta_f} \sinh^4\theta\, d\theta = \frac{3}{5}\frac{1}{4\pi}\frac{GM^2}{R^4}$$

2015

SM 3

$m = \frac{N_+ - N_-}{N}$, $E = -\frac{N}{2} J q\, m^2 - N\mu B m$, $Nm = N_+ - N_- = \overset{N-2N_-}{2N_+ - N}$

$\Rightarrow N_+ = \frac{N}{2}(1+m)$, $N_- = \frac{N}{2}(1-m)$

$$S = k \log \frac{N!}{N_+!(N-N_+)!} = k\left[N\log N - \cancel{N} - N_+\log N_+ + \cancel{N_+} - (N-N_+)\log(N-N_+) + \cancel{N} - \cancel{N_+}\right]$$

$$= k\left[N\log\frac{N}{N-N_+} - N_+\log\frac{N_+}{N-N_+}\right] = Nk\left[\log\frac{2}{1-m} - \frac{1+m}{2}\log\frac{1+m}{1-m}\right]$$

$$A = E - TS = -\frac{N}{2}Jq m^2 - N\mu B m - NkT\left[\log\frac{2}{1-m} - \frac{1+m}{2}\log\frac{1+m}{1-m}\right]$$

m small $\Rightarrow \log(1-m) \simeq -m - \frac{m^2}{2} - \frac{m^3}{3} - \dots$, $\log(1+m) = m - \frac{m^2}{2} + \frac{m^3}{3} - \dots$

$$A = N\left[-\frac{Jq}{2}m^2 - \mu B m - kT\log 2 + kT\left(-\cancel{m} - \frac{m^2}{2} - \cancel{\frac{m^3}{3}} - \frac{m^4}{4} - \dots\right) + kT\left(\cancel{m} + m^2 + \cancel{\frac{m^3}{3}} + \frac{m^4}{3} + \dots\right)\right]$$

$$\frac{1+m}{2}\left(m - \frac{m^2}{2} + \frac{m^3}{3} - \frac{m^4}{4} + \dots + m + \frac{m^2}{2} + \frac{m^3}{3} + \frac{m^4}{4} + \dots\right) = (1+m)\left(m + \frac{m^3}{3} + \dots\right)$$

$$= \left(m + m^2 + \frac{m^3}{3} + \frac{m^4}{3} + \dots\right)$$

$$\frac{A}{N} = -kT\log 2 - \mu B m + \left(-\frac{Jq}{2} + \frac{kT}{2}\right)m^2 + \frac{kT}{12}m^4 + \dots$$, spont: $B=0$

$$-\frac{Jq}{2} + \frac{kT_c}{2} = 0 \Rightarrow T_c = \frac{Jq}{k}$$

SM 4

$$\frac{\partial}{\partial t}\operatorname{tr}\rho^2 = \operatorname{tr}\left(\rho\frac{d\rho}{dt} + \frac{d\rho}{dt}\rho\right) = -\frac{1}{i\hbar}\left(\operatorname{tr}\left(\underbrace{\rho[\rho,H] + [\rho,H]\rho}_{\rho^2 H - \rho H\rho + \rho H\rho - H\rho^2 = [\rho^2, H]}\right)\right)$$

$$\operatorname{tr}[\rho^2 H] = \sum_n \langle n|\rho^2 H - H\rho^2|n\rangle = \sum_n \langle n|\rho^2 E_n - E_n\rho^2|n\rangle = 0 \Rightarrow \frac{\partial}{\partial t}\operatorname{tr}\rho^2 = 0$$

eq: $\frac{d\rho}{dt} = 0 \Rightarrow [\rho, H] = 0 \Rightarrow \rho = \rho(H)$

SM 5

$$Q = \sum_{N_r=0}^{\infty} z^{N_r} Q_{N_r}, \quad Q_{N_r} = \sum_{\{n_s\}} e^{-\beta\sum_s n_s\epsilon_s}, \quad E = \sum_s n_s\epsilon_s, \quad N_r = \sum_s n_s$$

$$= \sum_{n_1=0}^{N}\sum_{n_2=0}^{N}\dots z^{n_1+n_2+\dots} e^{-\beta\sum_s n_s\epsilon_s}$$

$$= \prod_s \sum_{n_s=0}^{N}\left(ze^{-\beta\epsilon_s}\right)^{n_s} = \prod_s \frac{1-e^{\beta(\mu-\epsilon_s)(N+1)}}{1-e^{\beta(\mu-\epsilon_s)}}$$

$S_k = 1 + x + x^2 + \dots + x^k$

$S_k - 1 = xS_k - x^{k+1}$

$S_k(1-x) = 1 - x^{k+1}$

$$\langle n_i\rangle = -\frac{1}{\beta}\frac{\partial}{\partial\epsilon_i}\ln Q = -kT\frac{\partial}{\partial\epsilon_i}\sum_s\left[\ln\left(1-e^{\beta(\mu-\epsilon_s)(N+1)}\right) - \ln\left(1-e^{\beta(\mu-\epsilon_s)}\right)\right]$$

$$= -\cancel{kT}\left[\frac{\cancel{\beta}(N+1)e^{\beta(\mu-\epsilon_i)(N+1)}}{1-e^{\beta(\mu-\epsilon_i)(N+1)}} - \frac{\cancel{\beta}e^{\beta(\mu-\epsilon_i)}}{1-e^{\beta(\mu-\epsilon_i)}}\right]$$

$$\langle n\rangle = \frac{1}{e^{\beta(\epsilon-\mu)}-1} - \frac{N+1}{e^{\beta(\epsilon-\mu)(N+1)}-1}$$

$$N\to\infty \Rightarrow \langle n\rangle \to \frac{1}{e^{\beta(\epsilon-\mu)}-1}$$

$$N=1 \Rightarrow \langle n\rangle = \frac{1}{e^{\beta(\epsilon-\mu)}-1} - \frac{2}{\underbrace{e^{2\beta(\epsilon-\mu)}-1}_{(e^{\beta(\epsilon-\mu)}-1)(e^{\beta(\epsilon-\mu)}+1)}} = \frac{e^{\beta(\epsilon-\mu)}-1}{(e^{\beta(\epsilon-\mu)}-1)(e^{\beta(\epsilon-\mu)}+1)} = \frac{1}{e^{\beta(\epsilon-\mu)}+1}$$

$$\left(\beta = \frac{1}{kT}\right)$$

CM1 $L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) - mgz$, $x = r\cos\theta$, $y = r\sin\theta \Rightarrow \dot{x} = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta}$

$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) - mgz$, $z - \alpha r^4 = 0$, $\dot{y} = \dot{r}\sin\theta + r\cos\theta\,\dot{\theta}$

$d(z-\alpha r^4) = dz - 4\alpha r^3 dr = 0$

$$\frac{d}{dt}m\dot{r} - mr\dot{\theta}^2 = \lambda(-4\alpha r^3) \Rightarrow m\ddot{r} - \frac{l^2}{mr^3} = -4\alpha\lambda r^3$$

$$\frac{d}{dt}mr^2\dot{\theta} = 0 \Rightarrow mr^2\dot{\theta} = l \text{ const} \Rightarrow mr\dot{\theta}^2 = \frac{l^2}{mr^3}$$

$$\frac{d}{dt}m\dot{z} + mg = \lambda \Rightarrow m\ddot{z} + mg = \lambda$$

circular horizontal: $\dot{z} = 0$, $\dot{r} = 0$, $r = r_0$

$$-\frac{l^2}{mr^3} = -4\alpha\lambda r^3, \quad mg = \lambda \Rightarrow \frac{l^2}{mr^3} = 4\alpha mgr^3$$

$$r_0 = \left(\frac{l^2}{4\alpha m^2 g}\right)^{1/6}, \quad \dot{\theta} = \frac{l}{mr_0^2}$$

$$\lambda = N_z, \quad \lambda(-4\alpha r_0^3) = N_r \Rightarrow \vec{N} = \lambda\hat{z} - 4\alpha\lambda r_0^3\,\hat{r}$$

$$K = \frac{1}{2}mr_0^2\dot{\theta}^2 = \frac{l^2}{2mr_0^2}, \quad U = -mgz = -mg\alpha r_0^4$$

$$-\frac{1}{2}\sum_i \vec{F}_i\cdot\vec{x}_i = -\frac{1}{2}\Big(\underbrace{\lambda\alpha r_0^4 - 4\alpha\lambda r_0^3 r_0}_{\vec{N}\cdot\vec{r}} - \underbrace{\lambda\alpha r_0^4}_{-mgz}\Big) = 2\alpha\lambda r_0^4$$

but $K = \dfrac{4\alpha m^2 g r_0^6}{2mr_0^2} = 2\alpha\lambda r_0^4 = -\frac{1}{2}\sum_i \vec{F}_i\cdot\vec{x}_i$

2015 $r = r_0 + \delta r \Rightarrow \dot{r} = \delta\dot{r}, \quad \dot{z} = 0$

$$m\,\delta\ddot{r} - \frac{l^2}{m(r_0+\delta r)^3} = -4\alpha\lambda(r_0+\delta r)^3$$

$$m(r+\delta r)^2\dot{\theta} = l \quad , \quad \lambda = mg$$

$$m\,\delta\ddot{r} - \frac{l^2}{m r_0^3}\left(1 - 3\frac{\delta r}{r_0}\right) = -4\alpha m g r_0^3\left(1 + 3\frac{\delta r}{r_0}\right)$$

$$m\,\delta\ddot{r} + 3\left(\frac{l^2}{m r_0^4} + 4\alpha m g r_0^2\right)\delta r = 0$$

$$\delta\ddot{r} + 3\underbrace{\left(\frac{l^2}{m^2 r_0^4} + 4\alpha g r_0^2\right)}_{\omega^2}\delta r = 0$$

$$\Rightarrow \omega = \sqrt{3\left(\frac{l^2}{m^2 r_0^4} + 4\alpha g r_0^4\right)}$$

CM2 $U(r) = \alpha r^6 \quad , \quad L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \alpha r^6$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \quad , \quad p_\theta = m r^2\dot{\theta} \Rightarrow \dot{r} = \frac{p_r}{m}, \quad \dot{\theta} = \frac{p_\theta}{mr^2}$$

$$H = m\dot{r}^2 + mr^2\dot{\theta}^2 - L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \alpha r^6 = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \alpha r^6$$

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}, \quad \dot{\theta} = \frac{p_\theta}{mr^2}, \quad \dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{mr^3} - 6\alpha r^5, \quad \dot{p}_\theta = 0$$

$$H\left(r, \theta, \frac{\partial S}{\partial r}, \frac{\partial S}{\partial \theta}\right) + \frac{\partial S}{\partial t} = \frac{1}{2m}\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{2mr^2}\left(\frac{\partial S}{\partial \theta}\right)^2 + \alpha r^6 + \frac{\partial S}{\partial t} = 0$$

$$S = S_r(r) + S_\theta(\theta) + S_t(t) = S_r(r) + \alpha_\theta\theta - \alpha_t t$$

$$\Rightarrow \left(\frac{\partial S_r}{\partial r}\right)^2 + \frac{\alpha_\theta^2}{r^2} + 2m\alpha r^6 - 2m\alpha_t = 0$$

$$S_r = \int dr\sqrt{2m(\alpha_t - \alpha r^6) - \frac{\alpha_\theta^2}{r^2}}$$

$$S = \int dr\sqrt{2m(\alpha_t - \alpha r^6) - \frac{\alpha_\theta^2}{r^2}} + \alpha_\theta\theta - \alpha_t t$$

$$\beta = \frac{\partial S}{\partial \alpha_\theta} = -\int\frac{\alpha_\theta\,dr}{r^2\sqrt{2m(\alpha_t - \alpha r^6) - \alpha_\theta^2/r^2}} + \theta$$

i.e. $\displaystyle\int\frac{\alpha_\theta\,dr}{r^2\sqrt{2m(\alpha_t - \alpha r^6) - \alpha_\theta^2/r^2}} = \theta - \beta$

$$J_r = \oint p_r dr \quad , \quad p_\theta = \alpha_\theta = 0 \Rightarrow p_r = \frac{\partial S}{\partial r} = \sqrt{2m(\alpha_t - \alpha r^6)}$$

$$J_r = \oint dr \sqrt{2m(E - \alpha r^6)} \qquad \alpha_t = E$$

adiabatic: $J_r \sim$ const

$$J_r = \sqrt{2mE} \oint dr \sqrt{1 - \frac{\alpha}{E} r^6} \quad , \quad x = \left(\frac{\alpha}{E}\right)^{1/6} r \Rightarrow dr = \left(\frac{E}{\alpha}\right)^{1/6} dx$$

$$= \sqrt{2mE} \left(\frac{E}{\alpha}\right)^{1/6} \oint dx \sqrt{1 - x^6}$$

$$= \sqrt{2m} \frac{E^{2/3}}{\alpha^{1/6}} 2\int_{-1}^{1} dx \sqrt{1-x^6} \Rightarrow E^{2/3} \sim \alpha^{1/6} \Rightarrow E \sim \alpha^{1/4}$$

CM3 $\quad H = \frac{1}{2I} l^2 + \alpha \sin\theta = \text{const.} \Rightarrow l^2 = 2I(H - \alpha \sin\theta) \qquad H_0 = \frac{l^2}{2I}$

$$J_0 = \oint \sqrt{2I(H - \alpha\sin\theta)}\, d\theta = \oint \sqrt{2IH_0}\, d\theta = 2\pi \sqrt{2IH_0}$$

$$\Rightarrow H_0 = \frac{1}{2I}\left(\frac{J_0}{2\pi}\right)^2 \quad , \quad \dot{w}_0 = \nu_0 = \frac{\partial H_0}{\partial J} = \frac{J_0}{(2\pi)^2 I} \Rightarrow w_0 = \frac{J_0 t}{(2\pi)^2 I}$$

$$H_1 = \alpha\sin\theta \quad , \quad \delta E = \overline{\delta H(w_0, J)} \quad , \quad \dot\theta = 2\pi\nu_0 \Rightarrow \theta = 2\pi w_0 = \frac{J_0}{2\pi I} t$$

$$\delta E = \overline{\alpha \sin\left(\frac{J}{2\pi I} t\right)} = 0 \Rightarrow \delta\nu = \frac{\partial\, \delta H}{\partial J} = 0$$

$$\frac{\partial Y_1}{\partial w_0} = \frac{\overline{H_1} - H_1}{\nu_0} = \frac{-\alpha\sin\theta}{\nu_0} = \frac{-\alpha \sin(2\pi w_0)\,(2\pi)^2 I}{J_0}$$

$$\Rightarrow Y_1 = \frac{2\pi\alpha I}{J} \cos(2\pi w_0) \quad , \quad Y = w_0 J - \frac{2\pi\alpha I}{J}\cos(2\pi w_0)$$

$$w = \frac{\partial Y}{\partial J} = w_0 + \frac{2\pi\alpha I}{J^2}\cos(2\pi w_0) = \theta/2\pi$$

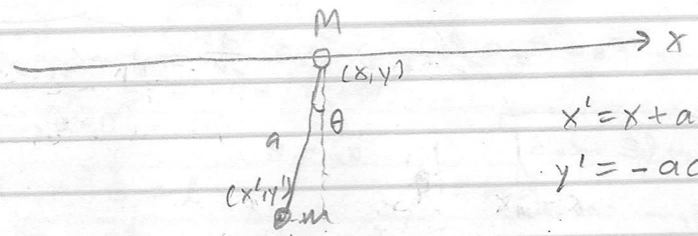
$$\Rightarrow \theta = 2\pi w_0 + \left(\frac{2\pi}{J}\right)^2 \alpha I \cos(2\pi w_0)$$

$$J_0 = \frac{\partial Y}{\partial w_0} = J + \frac{(2\pi)^2 \alpha I}{J} \sin(2\pi w_0) \Rightarrow J^2 - J_0 J + (2\pi)^2 \alpha I \sin(2\pi w_0) = 0$$

$$\Rightarrow J = \frac{J_0}{2} \pm \sqrt{\left(\frac{J_0}{2}\right)^2 - (2\pi)^2 \alpha I \sin(2\pi w_0)}$$

$$\theta = 2\pi w_0 + \frac{(2\pi)^2 \alpha I \cos(2\pi w_0)}{\left[\frac{J_0}{2} \pm \sqrt{\left(\frac{J_0}{2}\right)^2 - (2\pi)^2 \alpha I \sin(2\pi w_0)}\right]^2}$$

CM4



$x' = x + a\sin\theta$, $\dot{x}' = \dot{x} + a\cos\theta\,\dot{\theta}$

$y' = -a\cos\theta$, $\dot{y}' = a\sin\theta\,\dot{\theta}$

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}'^2 + \dot{y}'^2) - mgy'$$

$$= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + 2\dot{x}a\cos\theta\dot{\theta} + a^2\dot{\theta}^2\right) + mga\cos\theta$$

small θ: $\cos\theta \simeq 1 - \frac{1}{2}\theta^2 \to -\frac{1}{2}\theta^2$ (const. irrelevant)

$$L = \frac{M+m}{2}\dot{x}^2 + ma\cos\theta\dot{x}\dot{\theta} + \frac{ma^2}{2}\dot{\theta}^2 - \frac{mga}{2}\theta^2$$

$$\frac{d}{dt}\left[(M+m)\dot{x} + ma\cos\theta\dot{\theta}\right] = 0 \Rightarrow p = (M+m)\dot{x} + ma\left(1 - \frac{\theta^2}{2}\right)\dot{\theta} = const$$

$$\underbrace{\frac{d}{dt}\left[ma\cos\theta\dot{x} + ma^2\dot{\theta}\right]}_{\cancel{-ma\sin\theta\dot{\theta}\dot{x}} + ma\ddot{x}\cos\theta + ma^2\ddot{\theta}} + \cancel{ma\sin\theta\dot{x}\dot{\theta}} + mga\theta = 0$$

$$\dot{x} = \frac{p - ma\cos\theta\dot{\theta}}{M+m}$$

$$\ddot{x} = -\frac{ma}{M+m}(\cos\theta\ddot{\theta} - \sin\theta\dot{\theta}^2)$$

$$\Rightarrow ma\cos\theta\ddot{x} + ma^2\ddot{\theta} + mga\theta = 0,$$

$$(M+m)\ddot{x} + ma\cos\theta\ddot{\theta} - ma\sin\theta\dot{\theta}^2 = 0 \Rightarrow$$

$$\Rightarrow ma\cos\theta\left(-\frac{ma}{M+m}(\cos\theta\ddot{\theta} - \sin\theta\dot{\theta}^2)\right) + ma^2\ddot{\theta} + mag\theta = 0$$

$$\left(ma^2 - \frac{m^2a^2}{M+m}\cos^2\theta\right)\ddot{\theta} + \frac{m^2a^2}{M+m}\sin\theta\cos\theta\dot{\theta}^2 + mag\theta = 0$$

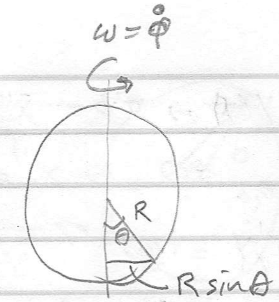
drop $\theta^2, \dot{\theta}^2 \Rightarrow \cos^2\theta \simeq 1 - \theta^2 \simeq 1$, $\sin\theta\cos\theta \simeq \theta$, $\dot{\theta}^2 \simeq 0$

$$ma^2\left(1 - \frac{m}{M+m}\right)\ddot{\theta} + mga\theta = 0$$

$$\ddot{\theta} + \frac{g}{a}\frac{1}{1 - \frac{m}{M+m}}\theta = 0$$

$$\omega = \sqrt{\frac{g}{a}\frac{1}{1 - \frac{m}{M+m}}}$$

CM 5



$$T = \tfrac{1}{2} mR^2\dot\theta^2 + \tfrac{1}{2} mR^2 \sin^2\theta\,\dot\varphi^2$$

$$V = R(1-\cos\theta)\,mg$$

$$L = \tfrac{1}{2} mR^2\dot\theta^2 + \tfrac{1}{2} mR^2\omega^2 \sin^2\theta - mgR(1-\cos\theta)$$

$$V_{eff} = mgR(1-\cos\theta) - \tfrac{1}{2} mR^2\omega^2\sin^2\theta$$

$$0 = \frac{\partial V_{eff}}{\partial\theta} = mgR\sin\theta - mR^2\omega^2\sin\theta\cos\theta = mgR\sin\theta\left(1 - \frac{R}{g}\omega^2\cos\theta\right)$$

$$\Rightarrow \sin\theta = 0 \quad \text{or} \quad \cos\theta = \frac{g}{R\omega^2}$$

$$\Rightarrow \theta = 0,\ \theta = \pi,\ \theta = \cos^{-1}\left(\frac{g}{R\omega^2}\right) \quad \left(\text{requires } g < R\omega^2 \text{ i.e. } \omega > \sqrt{g/R}\right)$$

$$(-\text{height} = R\cos\theta = g/\omega^2)$$

$$\frac{\partial^2 V_{eff}}{\partial\theta^2} = mgR\cos\theta - mR^2\omega^2(\cos^2\theta - \sin^2\theta)$$

$$\theta = 0: \quad \frac{\partial^2 V_{eff}}{\partial\theta^2} = mgR - mR^2\omega^2 = mgR\left(1 - \frac{R\omega^2}{g}\right)$$

stable for $\omega < \sqrt{g/R}$, unstable for $\omega > \sqrt{g/R}$

$$\theta = \pi: \quad \frac{\partial^2 V_{eff}}{\partial\theta^2} = -mgR - mR^2\omega^2 \Rightarrow \text{always unstable}$$

$$\theta = \cos^{-1}\left(\frac{g}{R\omega^2}\right): \quad \frac{d^2V_{eff}}{d\theta^2} = \frac{mg^2}{\omega^2} - m\omega^2R^2\left(\frac{2g^2}{R^2\omega^4} - 1\right)$$

$$= \frac{mg^2}{\omega^2} - \frac{2mg^2}{\omega^2} + m\omega^2R^2 = m\omega^2R^2 - \frac{mg^2}{\omega^2}$$

stable for $m\omega^2R^2 > \frac{mg^2}{\omega^2} \Rightarrow \omega^4 > \frac{g^2}{R^2} \Rightarrow \omega > \sqrt{\frac{g}{R}}$

i.e. exists (separate from $\theta = 0$) and is stable for $\omega > \sqrt{g/R}$

Consider $\theta = 0$ for $\omega < \sqrt{g/R}$. $0 = mR^2\ddot\theta - mR^2\omega^2\sin\theta\cos\theta + mgR\sin\theta$, $\theta \ll 1$

$$mR^2\ddot\theta + (mgR - mR^2\omega^2)\theta = 0 \quad \Rightarrow \quad \ddot\theta + \left(\frac{g}{R} - \omega^2\right)\theta = 0$$

$$\omega_{osc} = \sqrt{\frac{g}{R} - \omega^2} \quad \Rightarrow \quad T = \frac{2\pi}{\omega_{osc}} = \frac{2\pi}{\sqrt{\frac{g}{R} - \omega^2}}$$

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EM1-1 $\nabla^2\phi = 0$, $\phi = X(x)Y(y) \Rightarrow \nabla^2\phi = X''Y + XY'' = 0 \Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = \alpha$

$\phi|_{y=0} = \phi_0$, $\phi|_{x=0,a} = 0 \Rightarrow X'' = \alpha X \Rightarrow X \propto \sin\left(\frac{2\pi n}{a}x\right)$, $\alpha = \frac{\pi}{a}$

$Y'' = -\frac{n\pi}{a}Y \Rightarrow Y = e^{-\frac{\pi}{a}ny}$ (should go to 0)

$$\phi = \sum_{n=1}^{\infty} A_n \sin\left(\frac{\pi n}{a}x\right) e^{-\frac{\pi n}{a}y}$$

$$\phi|_{y=0} = \sum_{n=1}^{\infty} A_n \sin\left(\frac{\pi n}{a}x\right) = \phi_0 \quad \cdot \int_0^a \sin\left(\frac{\pi m}{a}x\right)dx$$

$$\frac{a}{2}A_m = \int_0^a \phi_0 \sin\left(\frac{\pi m}{a}x\right)dx = -\phi_0 \frac{a}{\pi m} \underbrace{\cos\left(\frac{\pi m}{a}x\right)\Big|_0^a}_{(-1)^m - 1}$$

$$\Rightarrow A_m = \phi_0 \frac{2}{\pi m} \underbrace{\left(1-(-1)^m\right)}_{2,\ m \text{ odd};\ 0,\ m \text{ even}} \Rightarrow m \to 2n+1$$

$$\phi = \frac{2}{\pi}\phi_0 \sum_{n=0}^{\infty} \frac{\sin((2n+1)\pi x/a)\, e^{-(2n+1)\pi y/a}}{2n+1}$$

$y \gg a \Rightarrow n=0$ dominates

$$\phi|_{y\gg a} = \frac{2}{\pi}\phi_0 \sin\left(\frac{\pi}{a}x\right) e^{-\frac{\pi}{a}y}$$

EM1-2 $\phi|_{r\le R} = 0 \Rightarrow \phi_s = \sum_{n=1}^{\infty} \beta_n \left(\frac{r}{R}\right)^{-n-1} P_n(\cos\theta)$

$$\phi|_{r=R} = \sum_{n=1}^{\infty} (\alpha_n + \beta_n) P_n(\cos\theta) = 0 \Rightarrow \alpha_n = -\beta_n$$

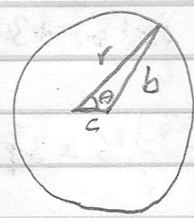
$$\Rightarrow \phi_s = -\sum_{n=1}^{\infty} \alpha_n \left(\frac{r}{R}\right)^{-n-1} P_n(\cos\theta)$$

$$\phi = \sum_{n=1}^{\infty} \alpha_n \left[\left(\frac{r}{R}\right)^n - \left(\frac{r}{R}\right)^{-n-1}\right] P_n(\cos\theta)$$

$$\sigma = -\epsilon_0 \frac{d\phi}{dr}\Big|_{r=R} = -\epsilon_0 \sum_{n=1}^{\infty} \alpha_n \left[\frac{n}{R}\left(\frac{r}{R}\right)^{n-1} - \frac{-n-1}{R}\left(\frac{r}{R}\right)^{-n-2}\right] P_n(\cos\theta)$$

$$= -\epsilon_0 \sum_{n=1}^{\infty} \alpha_n \frac{2n+1}{R} P_n(\cos\theta)$$

EMI-3 $b^2 = r^2 + c^2 - 2rc\cos\theta$



$c \ll b$

$r = c\cos\theta \pm \sqrt{c^2\cos^2\theta - c^2 + b^2}$

$r = c\cos\theta + b\sqrt{1 - \left(\frac{c}{b}\right)^2 \sin^2\theta}$

^ neglect

$r \simeq c\cos\theta + b = b\left(1 + \frac{c}{b}\cos\theta\right)$

$$\Phi = \sum_{l=0}^{1} \left(A_l r^l + B_l r^{-l-1}\right) P_l(\cos\theta) = A_0 + \frac{B_0}{r} + A_1 r\cos\theta + \frac{B_1\cos\theta}{r^2}$$

$$0 = \Phi\Big|_{r=a} = A_0 + \frac{B_0}{a} + A_1 a\cos\theta + \frac{B_1\cos\theta}{a^2}$$

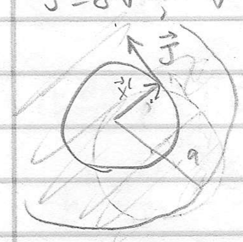
$$0 = \Phi\Big|_{r=b(1-\frac{c}{b}\cos\theta)} = A_0 + \frac{B_0}{b(1-\frac{c}{b}\cos\theta)} + A_1 b\left(1 - \frac{c}{b}\cos\theta\right) + \frac{B_1\cos\theta}{b^2(1-\frac{c}{b}\cos\theta)^2}$$

$$\sigma = -\epsilon_0 \frac{\partial\Phi}{\partial r}\Big|_{r=a} = -\epsilon_0\left(-\frac{B_0}{a^2} + A_1\cos\theta - \frac{2B_1\cos\theta}{a^2}\right)$$

$$Q = \oint \sigma\, dA = -\int_{-1}^{1} 2\pi a^2 d\mu\, (-\epsilon_0)\left(-\frac{B_0}{a^2} + A_1\mu - \frac{2B_1\mu}{a^2}\right)$$

$$= 2\pi a^2 \epsilon_0 \left(-\frac{2B_0}{a^2}\right) = -4\pi\epsilon_0 B_0 \Rightarrow B_0 = \frac{-Q}{4\pi\epsilon_0}$$

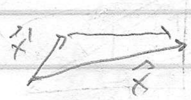
EMI-4 $\vec{J} = \sigma\vec{v}$, $\vec{v} = \omega(-\hat{x}y' + \hat{y}x') \Rightarrow \vec{J} = \sigma\omega(-\hat{x}y' + \hat{y}x')$



$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}\times(\vec{x}-\vec{x}')\, d^2x'}{|\vec{x}-\vec{x}'|^3}$$

$$= \frac{\mu_0\sigma\omega}{4\pi}\int d^2x' \frac{(-y'\hat{x} + x'\hat{y}) \times \left((x-x')\hat{x} + (y-y')\hat{y} + z\hat{z}\right)}{|\vec{x}-\vec{x}'|^3}$$

$$= \frac{\mu_0\sigma\omega}{4\pi}\int d^2x' \frac{\left(-y'(y-y') - x'(x-x')\right)\hat{z} + z\left(x'\hat{x} + y'\hat{y}\right)}{|\vec{x}-\vec{x}'|^3}$$



$|\vec{x}-\vec{x}'|^2 = r^2 + r'^2 - 2\vec{x}\cdot\vec{x}' \Rightarrow |\vec{x}-\vec{x}'|^{-3} = r^{-3}\left(1 - 2\frac{\vec{x}\cdot\vec{x}'}{r^2} + \left(\frac{r'}{r}\right)^2\right)^{-3/2}$

$$\vec{B} \approx \frac{\mu_0\sigma\omega}{4\pi r^3}\int d^2x' \left[\left(r'^2 - \vec{x}\cdot\vec{x}'\right)\hat{z} + z\vec{x}'\right]\left(1 + 3\frac{\vec{x}\cdot\vec{x}'}{r^2}\right)$$

$$\simeq \frac{\mu_0\sigma\omega}{4\pi r^3}\int d^2x' \left\{ r'^2\hat{z} + 3\frac{\vec{x}\cdot\vec{x}'}{r^2}\left[z\vec{x}' - \vec{x}\cdot\vec{x}'\hat{z}\right]\right\}$$

2015

$$\vec{B} \simeq \frac{\mu_0 \sigma \omega}{4\pi r^3} \int r' dr' d\theta \left\{ r'^2 \hat{z} + 3 \frac{x r' \cos\theta + y r' \sin\theta}{r^2} \left[z r' (\hat{x} \cos\theta + \hat{y} \sin\theta) - r' (x \cos\theta + y \sin\theta) \hat{z} \right] \right\}$$

$$\simeq \frac{\mu_0 \sigma \omega}{4\pi r^3} \int_0^a r' dr' \left\{ 2\pi r'^2 \hat{z} + 3\pi \frac{r'^2}{r^2} \left[x (z\hat{x} - x\hat{z}) + y (z\hat{y} - y\hat{z}) \right] \right\}$$

$$\simeq \frac{\mu_0 \sigma \omega}{4\pi r^3} \frac{a^4}{4} \pi \left\{ \left(2 - 3 \frac{x^2+y^2}{r^2}\right) \hat{z} + 3 \frac{xz}{r^2} \hat{x} + 3 \frac{yz}{r^2} \hat{y} \right\}$$

$$\simeq \frac{\mu_0 \sigma \omega}{16} \frac{a^4}{r^5} \left[3xz\hat{x} + 3yz\hat{y} + (2z^2 - x^2 - y^2) \hat{z} \right]$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \int d^2r' \, \sigma \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \simeq \frac{\sigma}{4\pi\epsilon_0} \int d^2r' \frac{\vec{r}}{r^3} = \frac{\sigma a^2}{4\epsilon_0} \frac{\vec{r}}{r^3}$$

EM1-5 $\quad \nabla^2 \phi - m\phi = 0, \quad \phi = X$

$$X''YZ + XY''Z + XYZ'' = mXYZ \Rightarrow \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = m$$

$$k_x^2 + k_y^2 + k_z^2 = m$$

$$X'' = k_x^2 X \Rightarrow X \propto e^{k_x x}, \text{ etc.}$$

$$\phi = A e^{\vec{k} \cdot \vec{x}}, \quad |\vec{k}| = \sqrt{m} \Rightarrow \phi = \int_S d^3k \, A_k e^{\vec{k} \cdot \vec{x}},$$

$$S = \left\{ \vec{k} \in \mathbb{C}^3 \mid k_x^2 + k_y^2 + k_z^2 = m \right\}$$

$$\phi = \phi_1 - \phi_2, \quad \phi|_b = 0, \quad \nabla^2\phi - m\phi = 0$$

$$\underbrace{\int_V \phi \nabla^2 \phi \, dv}_{m \int_V \phi^2 dv} = -\int_V (\nabla\phi)^2 dv + \underbrace{\oint_S \phi \nabla\phi \cdot d\vec{S}}_{0 \text{ by b.c.}}$$

$$\Rightarrow \int_V \left(m\phi^2 + (\nabla\phi)^2 \right) dv = 0$$

but $\phi^2 \geq 0$, $(\nabla\phi)^2 \geq 0 \quad m > 0 \Rightarrow \phi \equiv 0$

$$\Rightarrow \phi_1 \equiv \phi_2$$

No longer holds if $m < 0$ since $m\phi^2 + (\nabla\phi)^2$ can be < 0

QM1-1 nucleus $j_1=1$, e^- $j_2=\frac{1}{2}$ $\Rightarrow$ $j=\frac{1}{2},\frac{3}{2}$

$j_1\ j_2\ m_1 m_2$; $j\ m$

$\langle 1\tfrac{1}{2};1\tfrac{1}{2}|1\tfrac{1}{2};\tfrac{3}{2}\tfrac{3}{2}\rangle = 1$ (only possibility, +z saturated)

$\langle 1\tfrac{1}{2},0\tfrac{1}{2}|J_-|1\tfrac{1}{2},\tfrac{3}{2}\tfrac{3}{2}\rangle = \sqrt{3}\hbar\langle 1\tfrac{1}{2},0\tfrac{1}{2}|1\tfrac{1}{2},\tfrac{3}{2}\tfrac{1}{2}\rangle$

$= \sqrt{2}\hbar\langle 1\tfrac{1}{2},1\tfrac{1}{2}|1\tfrac{1}{2},\tfrac{3}{2}\tfrac{1}{2}\rangle = \sqrt{2}\hbar$

$\Rightarrow \langle 1\tfrac{1}{2},0\tfrac{1}{2}|1\tfrac{1}{2},\tfrac{3}{2}\tfrac{1}{2}\rangle = \sqrt{2/3}$

$\langle 1\tfrac{1}{2},1\,\tfrac{-1}{2}|J_-|1\tfrac{1}{2},\tfrac{3}{2}\tfrac{3}{2}\rangle = \sqrt{3}\hbar\langle 1\tfrac{1}{2},1\tfrac{-1}{2}|1\tfrac{1}{2},\tfrac{3}{2}\tfrac{1}{2}\rangle = 1$

$\Rightarrow \langle 1\tfrac{1}{2},1\tfrac{-1}{2}|1\tfrac{1}{2},\tfrac{3}{2}\tfrac{1}{2}\rangle = \frac{1}{\sqrt{3}}$

$\langle 1\tfrac{1}{2},1\tfrac{1}{2}|0\tfrac{1}{2},\tfrac{1}{2}\tfrac{1}{2}\rangle = 0$ $\left(m_1+m_2=\frac{3}{2}\neq m=1/2;\ j_1 \text{ differs too}\right)$

$\langle 1\tfrac{1}{2},1\tfrac{1}{2}|J_+|1\tfrac{1}{2},\tfrac{1}{2}\tfrac{1}{2}\rangle = 0$

$= \sqrt{2}\hbar\underbrace{\langle 1\tfrac{1}{2},0\tfrac{1}{2}|1\tfrac{1}{2},\tfrac{1}{2}\tfrac{1}{2}\rangle}_{\sqrt{\frac{1}{3}}} + \hbar\underbrace{\langle 1\tfrac{1}{2},1\tfrac{-1}{2}|1\tfrac{1}{2},\tfrac{1}{2}\tfrac{1}{2}\rangle}_{-\sqrt{\frac{2}{3}}}$

$\langle 1\tfrac{1}{2},-1\tfrac{-1}{2}|1\tfrac{1}{2},3/2\,\tfrac{-3}{2}\rangle = (-1)^{\frac{3}{2}-3/2} = 1$

$\langle 1\tfrac{1}{2},0\tfrac{-1}{2}|1\tfrac{1}{2},3/2\,\tfrac{-1}{2}\rangle = \sqrt{\tfrac{2}{3}}$

$\langle 1\tfrac{1}{2},-1\tfrac{1}{2}|1\tfrac{1}{2},\tfrac{3}{2}\tfrac{-1}{2}\rangle = \sqrt{\tfrac{1}{3}}$

$\langle 1\tfrac{1}{2},0\tfrac{-1}{2}|1\tfrac{1}{2},\tfrac{1}{2}\tfrac{-1}{2}\rangle = -\sqrt{\tfrac{1}{3}}$

$\langle 1\tfrac{1}{2},-1\tfrac{1}{2}|1\tfrac{1}{2},\tfrac{1}{2}\tfrac{-1}{2}\rangle = \sqrt{\tfrac{2}{3}}$

QM1-2 $i\hbar\frac{dA}{dt} = [A,H] = -igB$, $i\hbar\frac{dB}{dt} = [B,H] = igA$

$-\hbar^2\frac{d^2A}{dt^2} = -ig\,i\hbar\frac{dB}{dt} = g^2A \Rightarrow \frac{d^2A}{dt^2} = -\frac{g^2}{\hbar^2}A$, likewise B

$\Rightarrow \frac{d^2}{dt^2}\begin{matrix}\langle A\rangle\\ \langle B\rangle\end{matrix} = -\frac{g^2}{\hbar^2}\begin{matrix}\langle A\rangle\\ \langle B\rangle\end{matrix} \Rightarrow \langle A\rangle = \langle A\rangle_0\cos\left(\frac{gt}{\hbar}\right) - \langle B\rangle_0\sin\left(\frac{gt}{\hbar}\right)$

$\langle B\rangle = \langle B\rangle_0\cos\left(\frac{gt}{\hbar}\right) + \langle A\rangle_0\sin\left(\frac{gt}{\hbar}\right)$

$\frac{d}{dt}\langle A\rangle = -\frac{g}{\hbar}\langle B\rangle$

$\frac{d}{dt}\langle B\rangle = +\frac{g}{\hbar}\langle A\rangle$

2015 QM1-3

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi = E\psi \Rightarrow \psi \propto \sin(k\ x), \quad k = \frac{\sqrt{2mE}}{\hbar} = n\frac{\pi}{L}$$

$$\psi_n = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right), \quad \psi_0 = \sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right) = \langle x|0\rangle$$

expanded: $\psi_0' = \sqrt{\frac{1}{L}}\sin\left(\frac{\pi x}{2L}\right) = \langle x|0\rangle'$

$$\langle 0|0\rangle' = \int dx\, \langle 0|x\rangle\langle x|0\rangle' = \frac{\sqrt{2}}{L}\int_0^L dx\, \underbrace{\sin\left(\frac{\pi x}{L}\right)}_{2\sin\left(\frac{\pi x}{2L}\right)\cos\left(\frac{\pi x}{2L}\right)}\sin\left(\frac{\pi x}{2L}\right)$$

$$= \frac{2\sqrt{2}}{L}\int_0^L \sin^2\left(\frac{\pi x}{2L}\right)\cos\left(\frac{\pi x}{2L}\right)dx \qquad u = \sin\left(\frac{\pi x}{2L}\right), \quad du = \frac{\pi}{2L}\cos\left(\frac{\pi x}{2L}\right)dx$$

$$= \frac{4\sqrt{2}}{\pi}\int u^2\, du = \frac{4\sqrt{2}}{3\pi}\underbrace{\sin^3\left(\frac{\pi x}{2L}\right)}\Big|_0^L$$

$$= \frac{4\sqrt{2}}{3\pi} \qquad \sin^3\left(\frac{\pi}{2}\right) = 1$$

$$\text{Prob.} = |\langle 0|0\rangle'|^2 = \frac{32}{9\pi^2} \simeq 0.36$$

$$\langle p|0\rangle' = \int_0^{2L} dx\, \langle p|x\rangle\langle x|0\rangle', \quad \langle p|x\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{-ipx/\hbar}$$

$$= \frac{1}{\sqrt{2\pi\hbar}}\frac{1}{\sqrt{L}}\int_0^{2L} dx\, e^{-ipx/\hbar}\sin\left(\frac{\pi x}{2L}\right) = \frac{I}{\sqrt{2\pi\hbar L}}$$

$$I = -\frac{2L}{\pi}\int_0^{2L} e^{-ipx/\hbar}\, d\cos\left(\frac{\pi x}{2L}\right) = -\frac{2L}{\pi}\left[e^{-ipx/\hbar}\cos\left(\frac{\pi x}{2L}\right)\Big|_0^{2L} + \frac{ip}{\hbar}\int dx\, e^{-ipx/\hbar}\cos\left(\frac{\pi x}{2L}\right)\right]$$

$$= -\frac{2L}{\pi}\left[-e^{-2ipL/\hbar} - 1 + \frac{ip}{\hbar}\frac{2L}{\pi}\underbrace{\int_0^{2L} e^{-ipx/\hbar}\, d\sin\left(\frac{\pi x}{2L}\right)}_{\frac{ip}{\hbar}\int_0^{2L} e^{-ipx/\hbar}\sin\left(\frac{\pi x}{2L}\right)dx}\right]$$

$$I = \frac{2L}{\pi}\left(1 + e^{-2ipL/\hbar}\right) + \left(\frac{2L}{\pi}\right)^2\frac{p^2}{\hbar^2}I$$

$$I\left(1 - \left(\frac{2Lp}{\pi\hbar}\right)^2\right) = \frac{4L}{\pi}e^{-ipL/\hbar}\cos\left(\frac{pL}{\hbar}\right) \Rightarrow I = \frac{4L}{\pi}\frac{e^{-ipL/\hbar}\cos\left(\frac{pL}{\hbar}\right)}{1 - \left(\frac{2pL}{\pi\hbar}\right)^2}$$

$$|\langle p|0\rangle'|^2 = \frac{8L}{\pi^3\hbar}\frac{\cos^2\left(\frac{pL}{\hbar}\right)}{\left(1 - \left(\frac{2pL}{\pi\hbar}\right)^2\right)^2}$$

QM1-4 $D(R(\hat{e},\theta)) = e^{-i\hat{e}\cdot\vec{J}\theta/\hbar} = e^{-iJ_z\theta/\hbar}$

$j=0$: $\langle 0,0| e^{-iJ_z\theta/\hbar}|0,0\rangle = 1$

$j=\frac{1}{2}$: $\langle \frac{1}{2},\frac{1}{2}| e^{-iJ_z\theta/\hbar}|\frac{1}{2},\frac{1}{2}\rangle = e^{-i\theta/2}$

$\langle \frac{1}{2},\frac{1}{2}| D_z(\theta)|\frac{1}{2},-\frac{1}{2}\rangle = \langle \frac{1}{2},-\frac{1}{2}| D_z(\theta)|\frac{1}{2},\frac{1}{2}\rangle = 0$

$\langle \frac{1}{2},-\frac{1}{2}| D_z(\theta)|\frac{1}{2},-\frac{1}{2}\rangle = e^{i\theta/2}$

$j_1=j_2=\frac{1}{2}$: $\langle m_1, m_2| \ldots$

$\langle \frac{1}{2},\frac{1}{2}| D_z(\theta)|\frac{1}{2},\frac{1}{2}\rangle = e^{-i\theta}$

$\langle -\frac{1}{2},-\frac{1}{2}| D_z(\theta)|-\frac{1}{2},-\frac{1}{2}\rangle = e^{i\theta}$

$\langle \frac{1}{2},-\frac{1}{2}| D_z(\theta)|\frac{1}{2},-\frac{1}{2}\rangle = \langle -\frac{1}{2},\frac{1}{2}| D_z(\theta)|-\frac{1}{2},\frac{1}{2}\rangle = 1$

others 0

$$D_z^{(\frac{1}{2})}\otimes D_z^{(\frac{1}{2})} = \begin{array}{c|c} & \frac{1}{2}\frac{1}{2},\quad \frac{1}{2}\,-\frac{1}{2},\quad -\frac{1}{2}\,-\frac{1}{2},\quad -\frac{1}{2}\,\frac{1}{2} \\ \begin{matrix}\frac{1}{2}\frac{1}{2}\\ \frac{1}{2}-\frac{1}{2}\\ -\frac{1}{2}-\frac{1}{2}\\ -\frac{1}{2}\frac{1}{2}\end{matrix} & \begin{pmatrix} e^{-i\theta} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{i\theta} & 0 \\ 0 & 0 & 0 & \boxed{1} \end{pmatrix} \end{array}$$

$D_z^{(1)}(\theta)$ $\qquad$ $D_z^{(0)}(\theta)$

QM1-5 $H = \frac{p^2}{2m} + \frac{kx^2}{2} = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} = \hbar\omega\left(\frac{m\omega}{2\hbar}x^2 + \frac{1}{2m\omega\hbar}p^2\right)$

$a = \sqrt{\frac{m\omega}{2\hbar}}\,x + \frac{i}{\sqrt{2m\omega\hbar}}\,p$, $\quad a^\dagger a = \frac{m\omega}{2\hbar}x^2 + \frac{1}{2m\omega\hbar}p^2 + \underbrace{\frac{i}{2\hbar}\overbrace{[x,p]}^{i\hbar}}_{-1/2}$

$\Rightarrow H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$

note $[a,a^\dagger] = \frac{i}{2\hbar}\overbrace{\left(-[x,p]+[p,x]\right)}^{-2i\hbar} = 1$

Set $a^\dagger a|n\rangle = n|n\rangle$, note $a^\dagger a a^\dagger|n\rangle = a^\dagger(a^\dagger a+1)|n\rangle = (n+1)a^\dagger|n\rangle$

$a^\dagger a a|n\rangle = a(a^\dagger a-1)|n\rangle = (n-1)a|n\rangle$

$\Rightarrow a^\dagger|n\rangle = c_n|n+1\rangle$, $\quad a|n\rangle = c_n'|n-1\rangle$

$\langle n|a^\dagger a|n\rangle = |c_n'|^2 = n \Rightarrow c_n' = \sqrt{n}$, $\quad \langle n|aa^\dagger|n\rangle = |c_n|^2 = n+1 \Rightarrow$

$a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$, $\quad a|n\rangle = \sqrt{n}\,|n-1\rangle$ $\qquad a|0\rangle = 0$

$a^\dagger|0\rangle = |1\rangle$, $\quad a^\dagger|1\rangle = \sqrt{2}\,|2\rangle, \ldots$ $\quad (a^\dagger)^n|0\rangle = \sqrt{n!}\,|n\rangle \Rightarrow |n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle$

$H|n\rangle = \hbar\omega(n+\frac{1}{2})|n\rangle \Rightarrow E_n = \hbar\omega(n+\frac{1}{2})$

2015 EM11-1 $\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega(\omega+i\nu)}$ $\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$, $\nabla\times\vec{H} = \vec{J} + \frac{1}{c^2}\frac{\partial D}{\partial t}$

$\vec{D} = \epsilon(\omega)\vec{E}$, $\vec{B} = \mu_0\vec{H}$ $(\epsilon_0 = \mu_0 = 1)$

$i\vec{k}\times\vec{E} = i\omega\vec{B}$, $i\vec{k}\times\vec{B} = \vec{J} - i\omega\,\epsilon(\omega)\vec{E}$

$\vec{k}\times\vec{E} = \omega\vec{B}$, $\vec{k}\times\vec{B} = -\omega\left(1 - \frac{\omega_p^2}{\omega(\omega+i\nu)}\right)\vec{E}$

longitudinal: $\vec{k}\times\vec{E} = 0 \Rightarrow \vec{B} = 0$

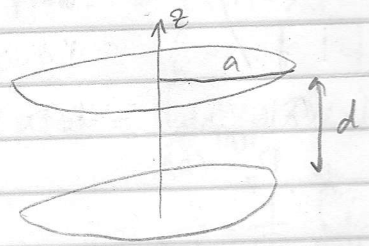
$$\Rightarrow 1 - \frac{\omega_p^2}{\omega(\omega+i\nu)} = 0 \Rightarrow \omega_p^2 = \omega(\omega+i\nu) = \omega^2 + i\nu\omega$$

$$\omega = \frac{-i\nu \pm \sqrt{4\omega_p^2 - \nu^2}}{2}$$

$$\vec{E} = \vec{E}_0 e^{i\vec{k}\cdot\vec{x}} e^{-i\omega t} = \vec{E}_0 e^{i\vec{k}\vec{x}} e^{-\frac{\nu}{2}t} e^{-i\sqrt{\omega_p^2 - \nu^2/4}\,t}$$

$\Rightarrow$ decays as $e^{-\nu/2\, t}$, oscillates with frequency $\omega_p\sqrt{1 - \frac{\nu^2}{4\omega_p^2}}$

EM11-2



$V = V_0\cos\omega t = Ed$

$d \ll a \ll c/\omega$

$\mu_0 = \epsilon_0 = 1$

symmetry $\Rightarrow$ quasistatic $\vec{E}$ vertical $\Rightarrow$ $\vec{E} \simeq \frac{V_0}{d}\hat{z}\cos\omega t$

$\nabla\times\vec{B} = \frac{d\vec{E}}{dt} = \frac{-V_0\omega}{d}\hat{z}\sin\omega t = \hat{z}(\partial_x B_y - \partial_y B_x)$ ← cylindrical coordinates

$\Rightarrow \vec{B} = \frac{V_0\omega}{2d}\sin\omega t\,(y\hat{x} - x\hat{y}) = -\frac{V_0\omega}{2d}\sin\omega t\; r\,\hat{\theta}$

$\sigma = \vec{E}\cdot(\pm\hat{z}) = \frac{V_0}{d}\cos\omega t \begin{cases} +, & \text{lower} \\ -, & \text{upper}\end{cases} \Rightarrow Q = \frac{\pi a^2}{d}V_0\cos\omega t \begin{cases} +, & \text{lower} \\ -, & \text{upper}\end{cases}$

wire: $I = \frac{dQ}{dt} = \frac{\pi a^2}{d}V_0\omega\sin\omega t$ ($-\hat{z}$ direction)

outside:

$2\pi r B = I = \frac{\pi a^2}{d}V_0\omega\sin\omega t \Rightarrow \vec{B} = -\frac{V_0\omega}{2d}\frac{a^2}{r}\sin\omega t\,\hat{\theta}$

$$(\mp)\underbrace{\hat{z}\times\hat{\theta}}_{-\hat{r}}\frac{V_0\omega}{2d}\sin\omega t\left(-\frac{a^2}{r} + r\right) = \vec{K}$$

$$\Rightarrow \vec{K} = \frac{V_0\omega}{2d}\left(r - \frac{a^2}{r}\right)\sin\omega t\,\hat{r} \begin{cases} +, & \text{lower} \\ -, & \text{upper}\end{cases}$$

EMII-3 initial $E = 2E_e$, $\vec{p} = 0$

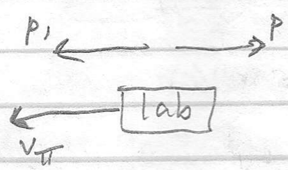
pions: $\vec{p} = 0 \Rightarrow p_{\pi 1} = p_{\pi 2} \Rightarrow E_{\pi 1} = E_{\pi 2} = E_\pi = E_e$

$$p_\pi = \sqrt{E_e^2 - m_\pi^2} = \gamma_\pi m_\pi v_\pi = E_\pi v_\pi = E_e v_\pi \Rightarrow v_\pi = \sqrt{1 - m_\pi^2/E_e^2}$$

$$\gamma_\pi = E_e/m_\pi$$

π rest frame: $E' = m_\pi = E_1' + E_2'$ $\quad (E_i' = p_i')$

$E_1' = E_2' = \frac{m_\pi}{2}$



boost to lab frame

$$\begin{pmatrix} \gamma_\pi & -\gamma_\pi v_\pi \\ -\gamma_\pi v_\pi & \gamma_\pi \end{pmatrix} \begin{pmatrix} E_1 \\ \pm E_1 \end{pmatrix} = \begin{pmatrix} \gamma_\pi E_1 \mp \gamma_\pi v_\pi E_1 \\ -\gamma_\pi v_\pi E_1 \pm \gamma_\pi E_1 \end{pmatrix}$$

$$= \gamma_\pi (1 \mp v_\pi) \begin{pmatrix} E_1 \\ E_1 \end{pmatrix}$$

$$E_{1,2} = \gamma_\pi (1 \mp v_\pi) E_{1,2}'$$

$$= E_e/m_\pi \left(1 \mp \sqrt{1 - m_\pi^2/E_e^2}\right) \frac{m_\pi}{2}$$

$$= \frac{1}{2}\left(E_e \mp \sqrt{E_e^2 - m_\pi^2}\right)$$

$m_\pi = 135\ \text{MeV}$, $E_e = 351\ \text{MeV} \Rightarrow E_{1,2} = \frac{1}{2}\left(351\ \text{MeV} \mp \sqrt{(351\ \text{MeV})^2 - (135\ \text{MeV})^2}\right)$

$\Rightarrow E_1 = 13.5\ \text{MeV}$, $E_2 = 337.5\ \text{MeV}$, $E_{1,2} = (175.5 \pm 162)\ \text{MeV}$

EMII-4 $E_x = B_x = 0$. $\vec{E}(x,y,z,t) = \vec{E}_0(y,z)\, e^{i(kx-\omega t)}$

$\vec{E}$ must be normal to inner wall.

inside (free space): $\nabla \cdot \vec{E} = 0$, $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -i\omega \vec{B}_0(y,z)\, e^{i(kx-\omega t)}$

no x component

but $\nabla \times \vec{E} = \hat{x}(\partial_y E_z - \partial_z E_y) - \hat{y}\,\partial_x E_z + \hat{z}\,\partial_x E_y$

$= \nabla \times \vec{E}_0\, e^{i(kx-\omega t)} - \hat{y}\, ik E_z + \hat{z}\, ik E_y$ $\quad \Rightarrow \nabla \times \vec{E}_0 = 0$

$\vec{E}_0 = \nabla \phi_0$, $\nabla^2 \phi_0 = \nabla \cdot \nabla \phi_0 = \nabla \cdot \vec{E}_0 = 0$

$\vec{E} = \nabla \phi_0$ normal to wall $\Rightarrow$ no tangential derivative of ϕ_0

$\Rightarrow \phi_0|_{\text{wall}} = A$ constant. $\nabla^2 \phi_0 = 0 \Rightarrow \phi_0 = A$ inside $\Rightarrow \vec{E} = 0$ inside

2015 EM11-5

$\beta_i = (n_i n_j + e_i^A e_j^A + e_i^B e_j^B)\beta_j$, $\vec{n}\times\vec{e}^A = \vec{e}^B$, $\vec{n}\times\vec{e}^B = -\vec{e}^A$

$\vec{n}\times\vec{\beta} = \vec{n}\times(\vec{n}(\vec{n}\cdot\vec{\beta}) + \vec{e}^A(\vec{e}^A\cdot\vec{\beta}) + \vec{e}^B(\vec{e}^B\cdot\vec{\beta})) = \vec{e}^B(\vec{e}^A\cdot\vec{\beta}) - \vec{e}^A(\vec{e}^B\cdot\vec{\beta})$

$\vec{n}\times(\vec{n}\times\vec{\beta}) = -\vec{e}^A(\vec{e}^A\cdot\vec{\beta}) - \vec{e}^B(\vec{e}^B\cdot\beta) = \vec{n}(\vec{\beta}\cdot\vec{n}) - \vec{\beta}$, $\vec{\beta} = \frac{1}{c}\frac{d\vec{x}}{dt}$

$$\frac{dW}{d\omega\, d\Omega} = \frac{e^2\omega^2}{4\pi^2 c}\sum_{m=A,B}\left|\vec{e}^m\cdot\underbrace{\int dt\,\vec{\beta}(t')e^{i\omega(t'-\vec{n}\cdot\vec{x}(t')/c)}}_{I}\right|^2$$

$$I = \int dt'\frac{\vec{\beta}(t')}{1-\vec{n}\cdot\vec{\beta}(t')}(1-\vec{n}\cdot\vec{\beta}(t'))e^{i\omega(t'-\vec{n}\cdot\vec{x}(t')/c)}$$

$$= \frac{i}{\omega}\int dt'\frac{d}{dt'}\left(\frac{\vec{\beta}(t')}{1-\vec{n}\cdot\vec{\beta}(t')}\right)e^{i\omega(t'-\vec{n}\cdot\vec{x}(t')/c)} , \quad \omega \text{ small}$$

$$\simeq \frac{i}{\omega}\frac{\vec{\beta}}{1-\vec{n}\cdot\vec{\beta}}\Big|_{-\infty}^{\infty} = \frac{i}{\omega}\left(\frac{\vec{\beta}''}{1-\vec{n}\cdot\vec{\beta}''} - \frac{\vec{\beta}'}{1-\vec{n}\cdot\vec{\beta}'}\right)$$

$$\frac{dW}{d\omega\, d\Omega} = \frac{e^2}{4\pi^2 c}\sum_{m=A,B}\left|\vec{e}^m\cdot\left(\frac{\vec{\beta}''}{1-\vec{n}\cdot\vec{\beta}''} - \frac{\vec{\beta}'}{1-\vec{n}\cdot\vec{\beta}'}\right)\right|^2$$

$\vec{e}^m$ polarization states

diff time, same polarization : coherent

diff polarization : incoherent

QMII-1 dist: $\psi_{nm}(x_1,x_2) = \psi_n(x_1)\psi_m(x_2) = \frac{2}{L}\sin\left(\frac{n\pi x_1}{L}\right)\sin\left(\frac{m\pi x_2}{L}\right)$

ground state ψ_{11}

triplet: sym. spin $\Rightarrow$ antisym. space

ground state $\psi_0 = \frac{1}{\sqrt{2}}(\psi_{12} - \psi_{21}) = \frac{\sqrt{2}}{L}\left(\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{2\pi x_2}{L}\right) - \sin\left(\frac{2\pi x_1}{L}\right)\sin\left(\frac{\pi x_2}{L}\right)\right)$

$$E_0 = \frac{\hbar^2}{2m}\left[\left(\frac{\pi}{L}\right)^2 + \left(\frac{2\pi}{L}\right)^2\right] = \frac{5\pi^2\hbar^2}{2mL}$$

singlet: antisym. spin $\Rightarrow$ sym. space

ground state $\psi_0 = \psi_{11} = \frac{2}{L}\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{\pi x_2}{L}\right)$

$H_1 = -\lambda\,\delta(x_2 - x_1)$, $\lambda > 0$

triplet: $E_1 = \int_0^L dx_1\,dx_2\,\psi_0^2 H_1 = -\lambda\frac{2}{L^2}\int_0^L dx\left[\left(\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi x}{L}\right) - \text{same}\right)\right]^2$

$= 0$

no effect because single particle wavefunctions don't overlap.

singlet: $E_1 = -\lambda\frac{4}{L^2}\int_0^L dx\,\sin^4\left(\frac{\pi x}{L}\right) < 0$;

decreases ground state energy because wavefunctions do overlap.

QMII-2 $i\hbar\partial_t U_I = V_I U_I \Rightarrow U_I(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^t V_I(t')\,dt' - \frac{1}{\hbar^2}\int_{t_0}^t dt'\int_{t_0}^{t'} dt''\,V_I(t')V_I(t'') + \dots$,

$c_{i\to n} = \langle n|U_I|i\rangle = \delta_{ni} - \frac{i}{\hbar}\int_{t_0}^t V(t')e^{i\omega_{ni}t'}dt' - \frac{1}{\hbar^2}\sum_m\int_{t_0}^t dt'\int_{t_0}^{t'} dt''\,V(t')e^{i\omega_{nm}t'}V(t'')e^{i\omega_{mi}t''}$

$= \delta_{ni} - \frac{i}{\hbar}V_0\int_{t_0}^t e^{(\eta+i\omega_{ni})t'}dt' - \frac{V_0^2}{\hbar^2}\sum_m\int_{t_0}^t dt'\int_{t_0}^{t'} dt''\,e^{(\eta+i\omega_{nm})t'}e^{(\eta+i\omega_{mi})t''}$

$c_{i\to i} = 1 - \frac{i}{\hbar}V_0\int_{t_0}^t e^{\eta t'}dt - \frac{V_0^2}{\hbar^2}\sum_m\int_{t_0}^t dt'\int_{t_0}^{t'} dt''\,e^{(\eta+i\omega_{im})t'}e^{(\eta-i\omega_{im})t''}$

$= 1 - \frac{i}{\hbar}\frac{V_0}{\eta}(e^{\eta t}-1) - \frac{V_0}{\hbar^2}\sum_m\int_{t_0}^t dt'\,e^{(\eta+i\omega_{im})t'}\frac{(e^{(\eta-i\omega_{im})t'}-1)}{\eta - i\omega_{im}}$ $\quad t_0 = 0$

$= 1 - \frac{i}{\hbar}\frac{V_0}{\eta}e^{\eta t} - \frac{V_0}{\hbar^2}\sum_m\left[\frac{e^{2\eta t}-1}{2\eta(\eta - i\omega_{im})} - \frac{e^{(\eta+i\omega_{im})t}-1}{(\eta-i\omega_{im})(\eta+i\omega_{im})}\right]$

$= 1 - \frac{i}{\hbar}\frac{V_0}{\eta}e^{\eta t} - \frac{V_0}{\hbar^2}\sum_m\frac{(\eta+i\omega_{im})(e^{2\eta t}-1) - 2\eta(e^{(\eta+i\omega_{im})t}-1)}{2\eta(\eta^2+\omega_{im}^2)}$

$= 1 - \frac{i}{\hbar}\frac{V_0}{\eta}e^{\eta t} - \frac{V_0}{2\eta\hbar^2}\sum_m\frac{\eta e^{2\eta t} + \eta + i\omega_{im}e^{2\eta t} - 2\eta e^{(\eta+i\omega_{im})t}}{\eta^2+\omega_{im}^2}$

$|c_{i\to i}|^2 = c_{i\to i}c_{i\to i}^* = (1 + \mathrm{Re}(c_{i\to i}-1))^2 + (\mathrm{Im}(c_{i\to i}))^2 \approx 1 + 2\,\mathrm{Re}(c_{i\to i}-1)$

2015 So $c^{(1)}_{i\to i} = -\frac{i}{\hbar}\frac{V_0}{\eta}e^{\eta t}$ does not contribute to order η (pure imaginary)

$$Re(c^{(2)}_{i\to i}) = -\frac{V_0}{2\eta\hbar^2}\sum_m \frac{\eta e^{2\eta t}+\eta-2\eta e^{\eta t}\cos(\omega_{im}t)}{\eta^2+\omega_{im}^2}$$

does contribute to order η.

(First order only contributes to order η^2)

1st order corresponds to interaction with V only once, i.e. at one time. 2nd order can interact at diff. times.

QMII-3 $V(r) = \begin{cases} \frac{kq}{r}, & r \geq R \\ \frac{kq}{R}, & r < R \end{cases} \to$ perturbation $V = \begin{cases} ke^2\left(\frac{1}{r}-\frac{1}{R}\right), & r<R \\ 0, & r\geq R\end{cases}$ $\sim\frac{1}{a_0}$

$\psi(r) = Ae^{-r/a_0}$, $1 = \int_0^\infty 4\pi r^2 dr\, A^2 e^{-2r/a_0} = 4\pi A^2\left[\frac{a_0}{2}\cdot 2\int_0^\infty r dr e^{-2r/a_0}\right]$

$$= 4\pi A^2 a_0\cdot\frac{a_0}{2}\int_0^\infty dr e^{-2r/a_0} = \pi A^2 a_0^3 \Rightarrow A = \frac{1}{\pi^{1/2}a_0^{3/2}}$$

$A^2 = \frac{1}{\pi a^3}$

$$E_1 = \int_0^R 4\pi r^2 dr\cdot\frac{1}{\pi a_0^3}\cdot e^{-2r/a_0}\cdot ke^2\left(\frac{1}{r}-\frac{1}{R}\right)$$

$$= \frac{4ke^2}{a_0^3R}\left[R\underbrace{\int_0^R r dr e^{-2r/a_0}}_{I_1} - \underbrace{\int_0^R r^2 dr e^{-2r/a_0}}_{I_2}\right]$$

$$I_2 = -\frac{a_0R^2}{2}e^{-2R/a_0} + a_0 I_1 = \frac{a_0}{4}\left[(-a_0^2-2a_0R-2R^2)e^{-2R/a_0}+a_0^2\right]$$

$$I_1 = -\frac{a_0R}{2}e^{-2R/a_0} - \frac{a_0^2}{4}\left(e^{-2R/a_0}-1\right) = \frac{a_0}{4}\left[(-a_0-2R)e^{-2R/a_0}+a_0\right]$$

$$E_1 = \frac{ke^2}{a_0^2R}\left[(-a_0R-2R^2+a_0^2+2a_0R+2R^2)e^{-2R/a_0}+a_0R-a_0^2\right]$$

$$= \frac{ke^2}{a_0^2R}\left[-a_0^2+a_0R + a_0^2(1+R/a_0)(1-2R/a_0+2R^2/a_0^2-\frac{4}{3}R^3/a_0^3)\right]$$

$$= \frac{ke^2}{R}\left[\cancel{-1}+R/a_0+\cancel{1}+R/a_0+(2-2)R^2/a_0^2+\left(-\frac{4}{3}+2\right)R^3/a_0^3\right]$$

$$= \frac{ke^2}{R}\cdot\frac{2}{3}R^3/a_0^3 = \frac{2}{3}\frac{ke^2R^2}{a_0^3}$$

QMII-4 $H_0 = \hbar \frac{b_0}{2}\sigma_z \quad V = \hbar b_1 \sigma_x \cos\omega t \quad \omega_{+-} = b_0$

$$V_I = e^{iH_0 t/\hbar} V e^{-iH_0 t/\hbar} \quad i\hbar \partial_t U_I(t,t_0) = V_I U_I(t,t_0)$$

$$\Rightarrow U_I(t,t_0) = 1 - \frac{i}{\hbar}\int_0^t dt'\, V_I(t') + \dots$$

$$\langle + | U_I(t) | - \rangle = -\frac{i}{\hbar}\int_0^t dt'\, e^{i\omega_{+-}t'} \langle + | \hbar b_1 \sigma_x | - \rangle \cos\omega t' \qquad \langle + | \sigma_x | - \rangle = (1\ 0)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = (1\ 0)\begin{pmatrix}1\\0\end{pmatrix} = 1$$

$$= -ib_1 \int_0^t \frac{1}{2}\left(e^{i(b_0+\omega)t'} + e^{i(b_0-\omega)t'}\right)$$

$$= -\frac{b_1}{2}\left[\frac{e^{i(b_0+\omega)t'}}{b_0+\omega} + \frac{e^{i(b_0-\omega)t'}}{b_0-\omega}\right]_0^t$$

$$= -\frac{b_1}{2}\left[\frac{e^{i(b_0+\omega)t}-1}{b_0+\omega} + \frac{e^{i(b_0-\omega)t}-1}{b_0-\omega}\right]$$

$$|\langle + | U_I(t) | - \rangle|^2 = \frac{b_1^2}{4}\left[\quad \dots \quad\right]\cdot\left[\frac{e^{-i(b_0+\omega)t}-1}{b_0+\omega} + \frac{e^{-i(b_0-\omega)t}-1}{b_0-\omega}\right]$$

$$= \frac{b_1^2}{4}\left[\frac{2-2\cos((b_0+\omega)t)}{(b_0+\omega)^2} + \frac{2-2\cos((b_0-\omega)t)}{(b_0-\omega)^2} + \frac{e^{2i\omega t} - 2e^{i\omega t}\cos(b_0 t) + 1}{b_0^2-\omega^2} + \frac{e^{-2i\omega t} - 2e^{-i\omega t}\cos(b_0 t) + 1}{b_0^2-\omega^2}\right]$$

$$= \frac{b_1^2}{2}\left[\frac{1-\cos[(b_0+\omega)t]}{(b_0+\omega)^2} + \frac{1-\cos[(b_0-\omega)t]}{(b_0-\omega)^2} + \frac{\cos(2\omega t) - 2\cos(\omega t)\cos(b_0 t) + 1}{(b_0-\omega)(b_0+\omega)}\right]$$

$$\simeq \frac{b_1^2}{2}\left[\frac{1-\cos[(b_0-\omega)t]}{(b_0-\omega)^2}\right] \quad \text{for } b_0-\omega \ll \omega$$

increases for time $\frac{\pi}{2(b_0-\omega)}$

rate $\frac{b_1^2}{2}\frac{\sin((b_0-\omega)t)}{b_0-\omega}$ not const.

Golden rule holds a) in continuous systems, where number of target states decreases in time due to uncertainty principle; b) for large t.

2015 QMII-5 $A=0$: ground state – spin singlet, $\psi(x_1,x_2) = \frac{2}{L}\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{\pi x_2}{L}\right)$

$$\Rightarrow |0\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right)|\psi\rangle, \quad \langle x_1,x_2|\psi\rangle = \frac{2}{L}\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{\pi x_2}{L}\right)$$

$$E_0 = \frac{\hbar^2}{2m}\left[\left(\frac{\pi}{L}\right)^2 + \left(\frac{\pi}{L}\right)^2\right] = \frac{\hbar^2\pi^2}{mL^2}$$

first excited: singlet $\otimes \frac{1}{\sqrt{2}}(\psi_{12}+\psi_{21})$

triplet $\otimes \frac{1}{\sqrt{2}}(\psi_{12}-\psi_{21})$

all with $E_1 = \frac{\hbar^2}{2m}\left(\left(\frac{\pi}{L}\right)^2 + \left(\frac{2\pi}{L}\right)^2\right) = \frac{5\hbar^2\pi^2}{2mL^2}$

$$\langle x|1\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle + |-+\rangle\right)\frac{\sqrt{2}}{L}\left(\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{2\pi x_2}{L}\right) - \sin\left(\frac{2\pi x_1}{L}\right)\sin\left(\frac{\pi x_2}{L}\right)\right)$$

$$\langle x|1^+\rangle = |++\rangle\frac{\sqrt{2}}{L}\left(\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{2\pi x_2}{L}\right) - \sin\left(\frac{2\pi x_1}{L}\right)\sin\left(\frac{\pi x_2}{L}\right)\right)$$

$$\langle x|1^-\rangle = |--\rangle\frac{\sqrt{2}}{L}\left(\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{2\pi x_2}{L}\right) - \sin\left(\frac{2\pi x_1}{L}\right)\sin\left(\frac{\pi x_2}{L}\right)\right)$$

$A>0$.

$$\langle 0|\underbrace{A\delta(x_1-x_2)\vec{S}_1\cdot\vec{S}_2}_{H'}|0\rangle = \frac{A}{2}\int dx_1 dx_2 \langle\psi|x_1,x_2\rangle\langle x_1,x_2|\psi\rangle$$

$$\cdot\,\delta(x_1-x_2)\left(\left(\langle+-| - \langle-+|\right)\vec{S}_1\cdot\vec{S}_2\left(|+-\rangle - |-+\rangle\right)\right)$$

$\vec{S}_1\cdot\vec{S}_2 = S_{1z}S_{2z} + S_{1x}S_{2x} + S_{1y}S_{2y}$, note $S_x|\pm\rangle = \frac{\hbar}{2}|\mp\rangle$

$S_y|\pm\rangle = \pm\frac{i\hbar}{2}|\mp\rangle$

since $\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$

$\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$

$\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = i\begin{pmatrix}0\\1\end{pmatrix}$

$\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = -i\begin{pmatrix}1\\0\end{pmatrix}$

$$\vec{S}_1\cdot\vec{S}_2|+-\rangle = -\frac{\hbar^2}{4}|+-\rangle + \frac{\hbar^2}{4}|-+\rangle + \frac{\hbar^2}{4}|-+\rangle$$

$$\vec{S}_1\cdot\vec{S}_2|-+\rangle = -\frac{\hbar^2}{4}|-+\rangle + \frac{\hbar^2}{4}|+-\rangle + \frac{\hbar^2}{4}|+-\rangle$$

$$\vec{S}_1\cdot\vec{S}_2\left(|+-\rangle - |-+\rangle\right) = -\frac{3}{4}\hbar^2|+-\rangle + \frac{3}{4}\hbar^2|-+\rangle$$

$$\left(\langle+-| - \langle-+|\right)\vec{S}_1\cdot\vec{S}_2\left(|+-\rangle - |-+\rangle\right) = -\frac{3}{4}\hbar^2 - \frac{3}{4}\hbar^2 = -\frac{3}{2}\hbar^2$$

$$\langle 0|H'|0\rangle = -\frac{3}{4}\hbar^2 A\int dx_1 \langle\psi|x_1x_1\rangle\langle x_1x_1|\psi\rangle$$

$$= -\frac{3}{4}\hbar^2 A\frac{4}{L^2}\int_0^L dx_1 \sin^4\left(\frac{\pi x_1}{L}\right), \quad x \equiv \frac{\pi x_1}{L} \Rightarrow dx_1 = \frac{L}{\pi}dx$$

$$= -\frac{3\hbar^2 A}{\pi L}\underbrace{\int_0^\pi dx \sin^4(x)}_{3\pi/8} = -\frac{9\hbar^2 A}{8L}$$