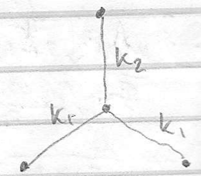
2014 CM1



spring k, l:

$(1+x)^{1/2} = 1 + x/2 - \frac{1}{8}x^2 + \cdots$

$(l')^2 = (l+x)^2 + y^2$

$= l^2 + 2lx + x^2 + y^2$

$U = \frac{1}{2}k(l'-l)^2 = \frac{1}{2}k(l'^2 - 2ll' + l^2)$

$= \frac{1}{2}k\left(2l^2 + 2lx + x^2 + y^2 - 2l^2\sqrt{1 + 2x/l + (\frac{x}{l})^2 + (\frac{y}{l})^2}\right)$

$= \frac{1}{2}kl^2\left(\cancel{2} + \cancel{2\frac{x}{l}} + (\frac{x}{l})^2 + \cancel{(\frac{y}{l})^2} - 2\left(\cancel{1} + \cancel{\frac{x}{l}} + \frac{1}{2}(\frac{x}{l})^2 + \frac{1}{2}(\frac{y}{l})^2 - \frac{4}{8}(\frac{x}{l})^2\right)\right)$

$= \frac{1}{2}kx^2$

$\uparrow y$

$\nearrow \alpha = \frac{\sqrt{3}}{2}x + \frac{1}{2}y$

$\searrow \beta = \frac{\sqrt{3}}{2}x - \frac{1}{2}y$

$U = \frac{1}{2}k_1(\alpha^2 + \beta^2) + \frac{1}{2}k_2 y^2$

$\alpha^2 + \beta^2 = \frac{3}{2}x^2 + \frac{1}{2}y^2 \Rightarrow U = \frac{3}{4}k_1 x^2 + \left(\frac{k_1}{4} + \frac{k_2}{2}\right)y^2$

$H = \frac{p_x^2 + p_y^2}{2m} + \frac{3}{4}k_1 x^2 + \left(\frac{k_1}{4} + \frac{k_2}{2}\right)y^2$

$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m}$, $\dot{y} = \frac{p_y}{m}$, $\dot{p}_x = -\frac{3}{2}k_1 x$, $\dot{p}_y = -\left(\frac{k_1}{2} + k_2\right)y$

$\Rightarrow \ddot{x} = -\frac{3}{2}\frac{k_1}{m}x$, $\ddot{y} = -\frac{1}{m}\left(\frac{k_1}{2} + k_2\right)y \Rightarrow \omega_1 = \sqrt{\frac{3}{2}\frac{k_1}{m}}$, $\omega_2 = \sqrt{\frac{k_1 + 2k_2}{2m}}$

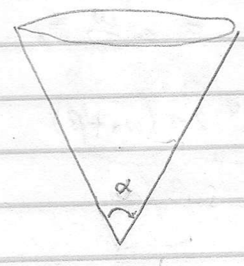
$f_1 = \omega_1/2\pi$ $\quad f_2 = \omega_2/2\pi$

$x_1 = A_1\sin(\omega_1 t) + B_1\cos(\omega_1 t)$

$x_2 = A_2\sin(\omega_2 t) + B_2\cos(\omega_2 t)$

$k_1 = k_2 \Rightarrow \omega_1 = \omega_2 \Rightarrow x_1, x_2$ linearly dependent

CM2



r, θ, φ : $\theta = \alpha/2$ constraint

$z = r\cos\theta \Rightarrow V = mgr\cos\theta = mgr\cos\alpha/2$

$x = r\sin\theta\cos\varphi \Rightarrow \dot{x} = \dot{r}\sin\theta\cos\varphi - r\sin\theta\sin\varphi\dot{\varphi}$

$y = r\sin\theta\sin\varphi \quad \dot{y} = \dot{r}\sin\theta\sin\varphi + r\sin\theta\cos\varphi\dot{\varphi}$

$z = r\cos\theta \quad \dot{z} = \dot{r}\cos\theta$

$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \dot{r}^2 + r^2\sin^2\theta\dot{\varphi}^2$

$L = \frac{m}{2}(\dot{r}^2 + r^2\sin^2\frac{\alpha}{2}\dot{\varphi}^2) - mgr\cos\alpha/2$, $p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$, $p_\varphi = mr^2\dot{\varphi}\sin^2\frac{\alpha}{2}$

$H = m\dot{r}^2 + mr^2\dot{\varphi}^2\sin^2\frac{\alpha}{2} - \frac{1}{2}m\dot{r}^2 - \frac{1}{2}mr^2\sin^2\frac{\alpha}{2}\dot{\varphi}^2 + mgr\cos\alpha/2$

$H = \frac{p_r^2}{2m} + \frac{p_\varphi^2}{2mr^2\sin^2\alpha/2} + mgr\cos\frac{\alpha}{2} = E$, $\quad p_\varphi = l$, $\dot{\varphi} = \frac{l}{mr^2\sin^2\alpha/2}$

$\dot{r} = p_r/m$

$J_r = \oint p_r dr$; $p_r = \sqrt{2m\left(E - \frac{l^2}{2mr^2\sin^2\alpha/2} - mgr\cos\frac{\alpha}{2}\right)}$

$J_r = \sqrt{2mE}\oint dr\sqrt{1 - \frac{l^2}{2mE\sin^2\alpha/2}\frac{1}{r^2} - \frac{mg\cos\alpha/2}{E}r}$, $x = r\cos\alpha/2 \Rightarrow r = \frac{x}{\cos\alpha/2}$

$dr = \frac{dx}{\cos\alpha/2}$

$$J_r = \sqrt{2mE}\,\frac{1}{\cos\alpha/2}\oint dx\sqrt{1-\frac{l^2}{2mE\tan^2\alpha/2}\frac{1}{x^2}-\frac{mg}{E}x}$$

$$\tan\theta \gg g m^{1/2} l/E^{3/2} \Rightarrow \frac{l^2}{\tan^2\theta} \ll \frac{E^3}{g^2 m} \Rightarrow \frac{l^2}{mE\tan^2\theta} \ll \left(\frac{E}{mg}\right)^2 \sim x^2$$

$$J_r \simeq \frac{\sqrt{2mE}}{\cos\alpha/2}\oint dx\sqrt{1-\frac{mg}{E}x} \qquad z = \frac{mg}{E}x$$

$$\simeq \frac{\sqrt{2mE}}{\cos\alpha/2}\frac{E}{mg}\oint dz\sqrt{1-z} = \text{const}$$

$$\Rightarrow E^{3/2} \sim \cos(\alpha/2) \Rightarrow E \sim \left[\cos(\alpha/2)\right]^{2/3}$$

CM3 $$H = mc^2\sqrt{1+\frac{p^2}{m^2c^2}} + \frac{m\omega^2x^2}{2} \simeq mc^2\left(1+\frac{p^2}{2m^2c^2}-\frac{p^4}{8m^4c^4}\right)+\frac{m\omega^2x^2}{2}$$

$$\simeq mc^2 + \frac{p^2}{2m} - \frac{p^4}{8m^3c^2} + \frac{m\omega^2x^2}{2}$$

$x = A\sin\omega t$

$p = m\omega A\cos\omega t = m\omega\sqrt{\frac{2E}{m\omega^2}}\cos\omega t = \sqrt{2mE}\cos\omega t$

$$H_0 = \frac{p^2}{2m} + \frac{m\omega^2x^2}{2} \quad (\text{ignore } mc^2) \qquad E = \tfrac{1}{2}m\omega^2A^2$$

$$J_0 = \oint p\,dx = \sqrt{2mE}\oint dx\sqrt{1-\frac{m\omega^2x^2}{2E}} \qquad p^2 = 2m\left(E-\frac{m\omega^2x^2}{2}\right)$$

$$\cos\theta = \sqrt{\frac{m}{2E}}\,\omega x \Rightarrow dx = \sqrt{\frac{2E}{m}}\frac{1}{\omega}\sin\theta\,d\theta$$

$$J_0 = \frac{-2E}{\omega}\oint \sin^2\theta\,d\theta = \frac{2\pi E}{\omega} \Rightarrow H_0 = \frac{\omega}{2\pi}J_0, \quad \nu_0 = \frac{\partial H}{\partial J_0} = \frac{\omega}{2\pi}$$

$$\Rightarrow w_0 = \frac{\omega}{2\pi}t - \beta, \quad \omega t = 2\pi(w_0+\beta)$$

$$H_1 = -\frac{p^4}{8m^3c^2} = -\frac{4m^2E^2}{8m^3c^2}\cos^4 2\pi(w_0+\beta), \qquad E = \frac{\omega}{2\pi}J_0$$

$$= -\frac{E^2}{2mc^2}\cos^4 2\pi(w_0+\beta) = \frac{-\omega^2J_0^2}{8\pi^2mc^2}\underbrace{\cos^4 2\pi(w_0+\beta)}_{3/8}$$

$$E_1 = -\frac{3}{64}\frac{\omega^2J_0^2}{\pi^2mc^2} = -\frac{3}{16}\frac{E^2}{mc^2}$$

$$\nu = \frac{\partial E}{\partial J} = \nu_0 + \frac{\partial E_1}{\partial J} \simeq \nu_0 - \frac{3}{32}\frac{\omega^2J_0}{\pi^2mc^2}$$

$$\Delta\nu = -\frac{3}{16\pi}\frac{\omega E}{mc^2} = -\frac{3}{8}\frac{\nu E}{mc^2} \Rightarrow \frac{\Delta\nu}{\nu} = -\frac{3}{8}\frac{E}{mc^2}$$

CM4 $H = \frac{p^2}{2m} + \frac{kx^4}{4} = E$, $\dot{x} = \frac{p}{m}$, $\dot{p} = -kx^3 \Rightarrow \ddot{x} = -\frac{k}{m}x^3$

$p = \sqrt{2mE}\sqrt{1 - \frac{k}{4E}x^4}$, $a = \pm\left(\frac{4E}{k}\right)^{1/4}$ (where $p=0$)

$\frac{dx}{dt} = \sqrt{\frac{2E}{m}}\sqrt{1-(x/a)^4}$, $E = \frac{k}{4}a^4$

$$T = \oint \frac{dt}{dx}dx = 4\int_0^a \frac{dt}{dx}dx = 4\sqrt{\frac{m}{2E}}\int_0^a \frac{dx}{\sqrt{1-(x/a)^4}}$$

$$= 4\sqrt{\frac{2m}{k}\frac{1}{a^2}}\ 1.31103\, a = 4\sqrt{\frac{2m}{k}}\ 1.31103/a$$

$a \to 0 \Rightarrow T \to \infty$ (slow oscillation due to short range freedom)

$a \to \infty \Rightarrow T \to 0$ (extremely fast oscillation due to strength of well at large distance)

CM5. picks up speed $-\frac{dL}{dt}$ per bounce (every $\Delta t = \frac{L}{v}$)

$\Rightarrow \frac{dv}{dt} = \frac{-v}{L}\frac{dL}{dt} \Rightarrow \frac{dv}{v} = \frac{-dL}{L} \Rightarrow \log v = -\log L + C \Rightarrow v = AL^{-1}$

$E = \frac{1}{2}mv^2 = \frac{1}{2}mA^2L^{-2}$, $E_0 = \frac{1}{2}mA^2L_0^{-2} \Rightarrow \frac{1}{2}mA^2 = E_0L_0^2$

$E = E_0\left(\frac{L_0}{L}\right)^2$

$A = \sqrt{\frac{2E_0L_0^2}{m}}$, $mv = \sqrt{2mE_0}\frac{L_0}{L}$

$\Delta p = m\frac{dL}{dt} + 2mv$ per bounce, $\frac{\Delta p}{\Delta t} = \left(m\frac{dL}{dt} + 2mv\right)\frac{v}{2L} = mv\frac{1}{L}\left(\frac{1}{2}\frac{dL}{dt} + mv\right)$

$$\bar{F} = \sqrt{2mE_0}\frac{L_0}{L^2}\left(\frac{1}{2}\frac{dL}{dt} + \sqrt{2mE_0}\frac{L_0}{L}\right)$$

$\frac{dL}{dt} \sim 0 \Rightarrow \bar{F} = 2mE_0\frac{L_0^2}{L^3} \Rightarrow \bar{F}L^3 = 2mE_0L_0^2$ constant

$\gamma = 3$

2014 SM1 harmonic oscillator (classical)

$$H = \sum_i^N \frac{p_i^2}{2m} + \frac{m\omega^2 x_i^2}{2} \le E$$

$$\Sigma(E) = \frac{1}{h^N}\int_{H\le E} d^N p\, d^N x \quad , \text{ let } P_i = \frac{p_i}{\sqrt{2m}} , \ Q_i = \sqrt{\frac{m}{2}}\,\omega x_i$$

$$\Rightarrow d^N p_i\, d^N x_i = (2m)^{N/2}\left(\frac{2}{m}\right)^{N/2}\omega^{-N} d^N P_i\, d^N Q_i$$

$$\Sigma(E) = \left(\frac{2}{h\omega}\right)^N \underbrace{\int_{H\le E} d^N P\, d^N Q}_{\text{volume of 2N-dimensional sphere of radius } \sqrt{E}}$$

$H = \Sigma_i (P_i^2 + Q_i^2)$ defines 2N-dim. sphere with radius $\sqrt{H}$

$$\Sigma(E) = \left(\frac{2}{h\omega}\right)^N \frac{\pi^N}{N!} E^N = \left(\frac{E}{\hbar\omega}\right)^N \frac{1}{N!}$$

$$S = k\log\Omega = k\log\Sigma = k\left(N\log\left(\frac{E}{\hbar\omega}\right) - N\log N + N\right)$$

$$= Nk\left[1 + \log\left(\frac{E}{N\hbar\omega}\right)\right]$$

SM2
$$Z = \sum_{k=1}^N e^{-\beta k \varepsilon} = \sum_{k=1}^N \left(e^{-\beta\varepsilon}\right)^k \quad , \text{ note } Z + 1 = e^{\beta\varepsilon} Z + e^{-\beta\varepsilon N}$$

$$Z(1 - e^{\beta\varepsilon}) = e^{-\beta\varepsilon N} - 1 \Rightarrow Z = \frac{1 - e^{-\beta\varepsilon N}}{e^{\beta\varepsilon} - 1}$$

$$\langle E\rangle = \frac{-\frac{\partial Z}{\partial\beta}}{Z} = -\frac{\partial}{\partial\beta}\log Z \quad , \ \log Z = \log(1 - e^{-\beta\varepsilon N}) - \log(e^{\beta\varepsilon} - 1)$$

$$= \frac{-\varepsilon N e^{-\beta\varepsilon N}}{1 - e^{-\beta\varepsilon N}} + \frac{\varepsilon e^{\beta\varepsilon}}{e^{\beta\varepsilon} - 1} \quad , \ \beta = \frac{1}{kT}$$

$$= \frac{-\varepsilon N}{e^{\beta\varepsilon N} - 1} + \frac{\varepsilon}{1 - e^{-\beta\varepsilon}}$$

$$T \to 0 \Rightarrow \beta \to \infty \Rightarrow \langle E\rangle \to \frac{-\varepsilon N}{e^{\infty} - 1} + \frac{\varepsilon}{1} = \varepsilon$$

$$T \to \infty \Rightarrow \beta \to 0 \Rightarrow \langle E\rangle \to \varepsilon\left(\frac{-N}{\beta\varepsilon N + \frac{1}{2}(\beta\varepsilon N)^2} + \frac{1}{\beta\varepsilon - \frac{1}{2}(\beta\varepsilon)^2}\right)$$

$$\langle E\rangle \to \varepsilon\left[\frac{-N}{\beta\varepsilon N(1 + \frac{1}{2}\beta\varepsilon N)} + \frac{1}{\beta\varepsilon(1 - \frac{1}{2}\beta\varepsilon)}\right] \simeq \varepsilon\left[-\frac{1}{\beta\varepsilon}(\cancel{1} - \tfrac{1}{2}\beta\varepsilon N) + \frac{1}{\beta\varepsilon}(\cancel{1} + \tfrac{1}{2}\beta\varepsilon)\right]$$

$$\langle E\rangle \to \frac{1}{2\beta}\left[\beta\varepsilon N + \beta\varepsilon\right] = \frac{N+1}{2}\varepsilon$$

SM3 $Q_N = \sum_{\{n_{|\varepsilon\rangle}\}}' g\{n_{|\varepsilon\rangle}\} \exp\left(-\beta \sum_{|\varepsilon\rangle} n_{|\varepsilon\rangle} \varepsilon\right)$, $'$: $\sum_{|\varepsilon\rangle} n_{|\varepsilon\rangle} = N$

$$Q = \sum_{N=1}^{\infty} z^N Q_N, \quad z = e^{\beta\mu}$$

$$= \left(\prod_{|\varepsilon\rangle} \sum_{n_{|\varepsilon\rangle}=0}^{\infty} \cdots\right) z^{\sum_{|\varepsilon\rangle} n_{|\varepsilon\rangle}} g\{\}e^{-\beta \sum_{|\varepsilon\rangle} n_{|\varepsilon\rangle} \varepsilon} = \prod_{|\varepsilon\rangle} \left[\sum_{n_{|\varepsilon\rangle}}' \exp\left(\beta(\mu-\varepsilon) n_{|\varepsilon\rangle}\right) \right]$$

s.t. $g\{n_{|\varepsilon\rangle}\} = 1$

$$\langle n_{|\varepsilon\rangle} \rangle = \frac{-1}{\beta} \frac{\frac{\partial Q}{\partial \varepsilon}}{Q} = \frac{-1}{\beta} \frac{d}{d\varepsilon} \log Q$$

$$\log Q = \sum_{|\varepsilon\rangle} \log \left(\sum_{n_{|\varepsilon\rangle}=0}^{2} e^{\beta(\mu-\varepsilon) n_{|\varepsilon\rangle}} \right)$$

$$\langle n_{|\varepsilon\rangle} \rangle = \frac{-1}{\beta} \frac{-\beta e^{\beta(\mu-\varepsilon)} - 2\beta e^{2\beta(\mu-\varepsilon)}}{1 + e^{\beta(\mu-\varepsilon)} + e^{2\beta(\mu-\varepsilon)}} = \frac{e^{\beta(\mu-\varepsilon)} + 2e^{2\beta(\mu-\varepsilon)}}{1 + e^{\beta(\mu-\varepsilon)} + e^{2\beta(\mu-\varepsilon)}}$$

Note $(1+x+x^2)(x-1) = x^3 - 1$

$$\langle n_{|\varepsilon\rangle} \rangle = \frac{\left(e^{\beta(\mu-\varepsilon)} + 2e^{2\beta(\mu-\varepsilon)}\right)\left(e^{\beta(\mu-\varepsilon)} - 1\right)}{e^{3\beta(\mu-\varepsilon)} - 1} = \frac{-e^{2\beta(\mu-\varepsilon)} + 2e^{3\beta(\mu-\varepsilon)} - e^{\beta(\mu-\varepsilon)}}{e^{3\beta(\mu-\varepsilon)} - 1}$$

$$= \frac{e^{\beta(\mu-\varepsilon)} + e^{2\beta(\mu-\varepsilon)} + e^{3\beta(\mu-\varepsilon)}}{1 - e^{3\beta(\mu-\varepsilon)}} - \frac{3e^{3\beta(\mu-\varepsilon)}}{1 - e^{3\beta(\mu-\varepsilon)}}$$

$$= \frac{e^{\beta(\mu-\varepsilon)} \cancel{\left(e^{3\beta(\mu-\varepsilon)} - 1\right)}}{-\cancel{\left(1 - e^{3\beta(\mu-\varepsilon)}\right)} \left(e^{\beta(\mu-\varepsilon)} - 1\right)} - \frac{3}{e^{3\beta(\varepsilon-\mu)} - 1}$$

$$= \frac{1}{e^{\beta(\varepsilon-\mu)} - 1} - \frac{3}{e^{3\beta(\varepsilon-\mu)} - 1}$$

2014 SM4 $\Sigma(\omega) = \underbrace{2}_{\text{longr. + trans.}} \frac{1}{h^2}\int d^2p\, d^2x$, $p = \hbar k = \frac{hk}{2\pi}$, $\int d^2x = L^2$

$\omega = ck$

$$\Sigma(\omega) = \frac{L^2}{2\pi^2}\int_0^k 2\pi k' dk' = \frac{L^2}{2\pi^2} c^{-2} \pi \omega^2 = \frac{c^{-2}L^2\omega^2}{2\pi}$$

$N = \Sigma(\omega_D) = \frac{c^{-2}L^2\omega_D^2}{2\pi}$ number of normal modes

$$D(\omega) = \frac{d\Sigma}{d\omega} = \frac{c^{-2}L^2}{\pi}\,\omega$$

$$U = \int_0^{\omega_D} U(\omega)\, D(\omega)\, d\omega$$

$$E(\omega) = \hbar\omega(n + \tfrac{1}{2})$$

$$U(\omega) = \frac{\hbar\omega}{2} + \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1}$$

one oscillator:

$$Q = \sum_{n=0}^{\infty} e^{-\beta\hbar\omega(n+\frac{1}{2})} = e^{-\beta\hbar\omega/2}\frac{1}{1-e^{-\beta\hbar\omega}}$$

$$U = \langle E\rangle = -\frac{\partial}{\partial\beta}\log Q ,$$

$$= -\frac{\partial}{\partial\beta}\left(-\frac{\beta\hbar\omega}{2} - \log(1-e^{-\beta\hbar\omega})\right)$$

$$= \frac{1}{2}\hbar\omega + \frac{\hbar\omega e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}}$$

$$= \frac{1}{2}\hbar\omega + \frac{\hbar\omega}{e^{\beta\hbar\omega}-1}$$

but $\beta\hbar\omega \sim \beta\hbar\omega_D >> 1$

$$\Rightarrow U(\omega) \simeq \hbar\omega\left(\frac{1}{2} + e^{-\beta\hbar\omega}\right)$$

$$U = \hbar\frac{c^2L^2}{\pi}\int_0^{\omega_D} d\omega\, \omega^2\left(\frac{1}{2} + e^{-\beta\hbar\omega}\right)$$

$$= \hbar\frac{c^2L^2}{\pi}\left(\frac{1}{6}\omega_D^3 + I\right)$$

$$I = \int_0^{\omega_D} d\omega\, \omega^2 e^{-\beta\hbar\omega} = \frac{2}{\beta\hbar}\int_0^{\omega_D} d\omega\, \omega\, e^{-\beta\hbar\omega} - \frac{\omega_D^2 e^{-\beta\hbar\omega_D}}{\beta\hbar}$$

$$= \frac{2}{(\beta\hbar)^2}\int_0^{\omega_D} e^{-\beta\hbar\omega}\, d\omega - \frac{\omega_D^2 e^{-\beta\hbar\omega_D}}{\beta\hbar} - \frac{2\omega_D}{(\beta\hbar)^2}e^{-\beta\hbar\omega_D}$$

$$= \frac{2}{(\beta\hbar)^3}\left[1 - \cancel{e^{-\beta\hbar\omega}}\right] - \cancel{\cdots} = \frac{2}{(\beta\hbar)^3}$$

$$U = \hbar\frac{c^2L^2}{\pi}\left(\frac{\omega_D^3}{6} + \frac{2}{(\beta\hbar)^3}\right) , \quad \beta = \frac{1}{kT} \Rightarrow \frac{\partial}{\partial T} = \frac{-1}{kT^2}\frac{\partial}{\partial\beta}$$

$$C_V = \frac{1}{kT^2}\hbar\frac{c^2L^2}{\pi}\cdot\frac{2}{\hbar^3}4(kT)^4 = \frac{8}{\pi}\frac{c^2k_B^3L^2}{\hbar^2}T^2$$

1-D: $\Sigma \sim \omega \Rightarrow D \sim 1 \Rightarrow U \sim T^2 \Rightarrow C_V \sim T$

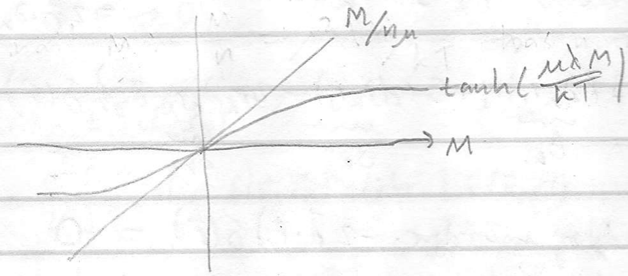
SMS $E = -\sum_{i=1}^{N} \mu_i B$, $Q = \sum_{\{\mu_i\}} e^{\beta \sum_i \sigma_i \mu B}$ $\sigma_i = \pm 1$ $(\mu_z = \sigma_i \mu)$

$$Q = \sum_{\sigma_1 = \pm 1} \cdots \sum_{\sigma_N = \pm 1} \prod_{i=1}^{N} e^{\beta \sigma_i \mu B} = \prod_{i=1}^{N} \sum_{\sigma_i = \pm 1} e^{-\sigma_i \beta \mu B} = \left(2\cosh(\beta \mu B)\right)^N$$

$$\langle \mu_z \rangle = \frac{1}{N\beta} \frac{\partial}{\partial B} \log Q = \frac{1}{\beta} \frac{\partial}{\partial B} \log\left(2\cosh(\beta \mu B)\right), \quad x = \beta \mu B = \frac{\mu}{kT}(H + \lambda M)$$

$$= \frac{1}{\beta} \frac{\sinh(x)}{\cosh(x)} \cdot \beta \mu = \mu \tanh x \quad , \quad \text{but } \langle \mu_z \rangle = M/n$$

i.e. $\frac{M}{n} = \mu \tanh(\mu \lambda M / kT)$ for $H = 0$



$\frac{M}{n\mu} = \tanh\left(\frac{\mu \lambda M}{kT}\right)$
has nonzero solutions
when $\frac{1}{n\mu} < \frac{\mu \lambda}{kT}$

$\Rightarrow T < \frac{n \lambda \mu^2}{k} \equiv T_c$

$$\frac{\partial M}{\partial H} = \frac{\partial}{\partial H} n \mu \tanh\left(\frac{\mu}{kT}(H + \lambda M)\right)$$

$$= n\mu^2/kT \, \operatorname{sech}^2\left(\frac{\mu}{kT}(H + \lambda M)\right)\left(1 + \lambda \frac{\partial M}{\partial H}\right)$$

take $M, H \to 0$

$$= \frac{T_c}{\lambda T}\left(1 + \lambda \frac{\partial M}{\partial H}\right) = \frac{T_c}{\lambda T} + \frac{T_c}{T} \frac{\partial M}{\partial H}$$

$$\Rightarrow \frac{\partial M}{\partial H}\left(1 - \frac{T_c}{T}\right) = \frac{T_c}{\lambda T}$$

$$\Rightarrow \frac{\partial M}{\partial H} = \frac{T_c}{\lambda T} \frac{1}{1 - T_c/T} = \frac{1}{\lambda} \frac{T_c}{T - T_c}$$

2014 EM1-1 $\rho = -\vec{d}\cdot\nabla\delta(\vec{r})$

$$Q = -\sum_i \int d^3r \; d_i \partial_i \delta(\vec{r}) = \sum_i \int d^3 r \,(\overbrace{\partial_i d_i}^{0})\, \delta(\vec{r}) = 0$$

$$p_j = -\int d^3r \; r_j d_i \partial_i \delta(\vec{r}) = \int d^3 r\, \partial_i (d_i r_j) \delta(\vec{r}) \quad , \quad \partial_i r_j = \delta_{ij}$$

$$= \int d^3 r \, d_j \, \delta(\vec{r}) = d_j$$

$$\Rightarrow \vec{p} = \vec{d}$$

$$Q_{ij} = -d_k \int d^3 r \,(3 r_i r_j - r_\ell r_\ell \delta_{ij})\, \partial_k \delta(\vec{r}) \qquad \begin{aligned}&\partial_k(\vec{r}\cdot\vec{r})\\ &= 2\vec{r}\cdot\partial_k\vec{r}\end{aligned}$$

$$= d_k \int d^3 r \left(3 \underbrace{(\partial_k r_i)}_{\delta_{ki}} r_j + 3 r_i \underbrace{(\partial_k r_j)}_{\delta_{kj}} - 2 r_\ell \underbrace{\partial_k r_\ell}_{\delta_{k\ell}} \delta_{ij}\right) \delta(\vec{r})$$

$$= d_k \int d^3 r \left(3 \; d_i r_j + 3 \; d_j r_i - 2\vec{d}\cdot\vec{r}\right)\delta(\vec{r}) = 0$$

$$p_{1m} = \int d^3 r \cdot r \sqrt{4\pi} \; Y^*_{1m}(\theta,\varphi)\, \rho(\vec{r}) \quad , \quad \text{take } \vec{d} = d\hat{z}$$

$$p_{1m} = -\vec{d}\cdot\sqrt{4\pi} \int d^3 r \cdot \underbrace{r \, Y^*_{1m}(\theta,\varphi)}\, \nabla \delta(\vec{r})$$

$\propto z$ for $m=0$

$x+iy$ for $m=-1$

$x-iy$ for $m=+1$

$$= +\vec{d}\cdot\sqrt{4\pi} \int d^3 r \cdot \nabla\left(r \, Y^*_{1m}(\theta,\varphi)\right) \delta(\vec{r})$$

$$\nabla z = \hat{z} \quad , \quad \nabla(x \pm iy) = \hat{x} \pm i\hat{y}$$

$$p_{10} = \vec{d}\cdot\hat{z} , \quad p_{11} = -\frac{\vec{d}}{\sqrt{2}}\cdot(\hat{x} - i\hat{y}) \quad , \quad p_{1-1} = \frac{\vec{d}}{\sqrt{2}}\cdot(\hat{x}+i\hat{y})$$

$$p_{10} = d_z \; , \quad p_{11} = -\frac{1}{\sqrt{2}}(d_x - i d_y) \quad , \quad p_{1-1} = \frac{1}{\sqrt{2}}(d_x + i d_y)$$

EM1-2 treat dipole at $\vec{r}$ as charge $\frac{-d_2}{a}$ at $\vec{r}$, $\frac{d_2}{a}$ at $\vec{r}+\vec{a}$ $\left(\vec{d}_2 = \frac{d_2}{a}\vec{a}\right)$

$$E = \vec{d}_1 \cdot \left(\frac{-d_2}{a}\frac{\vec{r}}{r^3} + \frac{d_2}{a}\frac{\vec{r}+\vec{a}}{|\vec{r}+\vec{a}|^3} \right)$$

$$|\vec{r}+\vec{a}|^3 = (r^2 + 2\vec{r}\cdot\vec{a} + a^2)^{3/2} \simeq r^3\left(1+2\frac{\vec{r}\cdot\vec{a}}{r^2}\right)^{3/2}$$

$$|\vec{r}+\vec{a}|^{-3} \simeq r^{-3}\left(1 - 3\frac{\vec{r}\cdot\vec{a}}{r^2}\right)$$

$$= \vec{d}_1 \cdot \left(\frac{d_2}{a}\frac{\vec{a}}{r^3} + \frac{d_2}{a}\frac{\vec{r}}{r^3}(-3)\frac{\vec{r}\cdot\vec{a}}{r^2} \right)$$

$$= \vec{d}_1 \cdot \left(\frac{\vec{d}_2}{r^3} - 3\frac{\vec{r}}{r^5}\vec{r}\cdot\vec{d}_2 \right) = \frac{\vec{d}_1\cdot\vec{d}_2}{r^3} - 3\frac{(\vec{d}_1\cdot\vec{r})(\vec{d}_2\cdot\vec{r})}{r^5}$$

$T \to \infty \Rightarrow$ random orientation, $\overline{\vec{d}_1\cdot\vec{d}_2} = 0$, $\overline{\vec{d}_1\cdot\vec{r}} = 0$, $\overline{\vec{d}_2\cdot\vec{r}} = 0$

$\Rightarrow E \to 0$

EM1-3 $\rho(\vec{r}) = e\,\delta^3(\vec{r}-\vec{r}_0)$

$$E = \frac{e^2}{2}\int d^3r\, d^3r'\, \delta^3(\vec{r}-\vec{r}_0)\,\delta^3(\vec{r}'-\vec{r}_0)\, G(\vec{r},\vec{r}')$$

remove self-energy

$$= \frac{e^2}{2}G(\vec{r}_0,\vec{r}_0) = \frac{e^2}{2}\left[\cancel{\frac{1}{|\vec{r}_0-\vec{r}_0|}} - \frac{\epsilon-1}{\epsilon+1}\frac{1}{|\vec{r}_0-\vec{r}_0''|}\right]$$

$$= -\frac{e^2}{2}\frac{\epsilon-1}{\epsilon+1}\frac{1}{z_0-(-z_0)} = -\frac{\epsilon-1}{\epsilon+1}\frac{e^2}{4z_0}$$

$$\vec{F} = -\nabla_0 E = \hat{z}\frac{\partial}{\partial z_0}\left(\frac{\epsilon-1}{\epsilon+1}\frac{e^2}{4z_0}\right) = -\frac{\epsilon-1}{\epsilon+1}\frac{e^2}{4z_0^2}\hat{z}$$

conductor: $\epsilon \to \infty$

$$\vec{E} = \frac{e(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} - \frac{e(\vec{r}-\vec{r}'')}{|\vec{r}-\vec{r}''|^3},$$

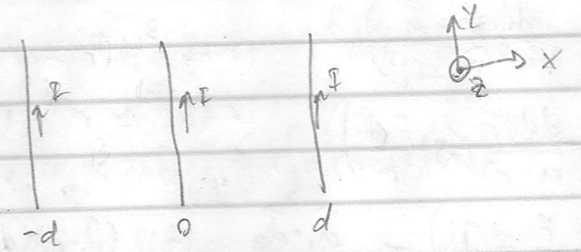
$z=0 \Rightarrow \vec{r}-\vec{r}' = x\hat{x}+y\hat{y}-z_0\hat{z}$

$\vec{r}-\vec{r}'' = x\hat{x}+y\hat{y}+z_0\hat{z}$

$$= \frac{-2z_0\hat{z}\,e}{(x^2+y^2+z_0^2)^{3/2}},\quad 4\pi\sigma = \vec{E}\cdot\hat{z} = \frac{-2z_0 e}{(x^2+y^2+z_0^2)^{3/2}}$$ at point $(x,y,0)$

2014 EM1-4 due to one wire $2\pi r B = \mu_0 I \Rightarrow \vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\varphi}$

$\hat{\varphi} = \hat{I} \times \hat{r}$



$$\vec{B} = \frac{\mu_0 I}{2\pi}\left(\frac{1}{x+d} + \frac{1}{x} + \frac{1}{x-d}\right)(-\hat{z}) = 0$$

$$\Rightarrow x(x-d) + (x+d)(x-d) + (x+d)x = 0$$

$$x^2 - xd + x^2 - d^2 + x^2 + dx = 3x^2 + d^2 = 0$$

$$\Rightarrow x = \pm\frac{d}{\sqrt{3}}$$

middle wire: x, z. $\vec{B} = \frac{\mu_0 I}{2\pi}\left(\frac{\hat{y}\times\hat{r}_-}{r_-} + \frac{\hat{y}\times\vec{r}_+}{r_+}\right)$, $\vec{F}/l = I\hat{y}\times\vec{B}$

$\vec{r}_\mp = (x \pm d)\hat{x} + z\hat{z}$, $\hat{I} = \hat{y} \Rightarrow \hat{y}\times\hat{r}_\mp = (x\pm d)(-\hat{z}) + z\hat{x}$

z, $d \pm x$ $\quad r_\mp^{-1} = (x^2 + d^2 \pm 2d\cdot x)^{-1/2} = d^{-1}\left(1 \pm \frac{2x}{d} + \frac{x^2}{d^2}\right)^{-1/2}$

$(1+x)^{-1/2} = 1 - \frac{x}{2} + \frac{3}{8}x^2 \cdots$

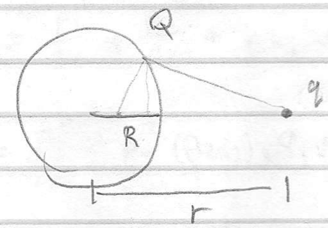
$$\vec{B} = \frac{\mu_0 I}{2\pi}\left(\frac{(x+d)(-\hat{z}) + z\hat{x}}{d\left(1+\frac{2x}{d}+\frac{x^2}{d^2}\right)^{1/2}} + \frac{(x-d)(-\hat{z}) + z\hat{x}}{d\left(1-\frac{2x}{d}+\frac{x^2}{d^2}\right)^{1/2}}\right)$$

$$\frac{\vec{F}}{l} = -\frac{\mu_0 I^2}{2\pi d}\left[\left[(x+d)\hat{x} + z\hat{z}\right]\left(1 - \frac{x}{d} + \underbrace{\left(\frac{-1}{2}+\frac{3}{2}\right)}_{1}\frac{x^2}{d^2}\right) + \left[(x-d)\hat{x} + z\hat{z}\right]\left(1 + \frac{x}{d} + \underbrace{\left(\frac{-1}{2}+\frac{3}{2}\right)}_{1}\frac{x^2}{d^2}\right)\right]$$

$$= -\frac{\mu_0 I^2}{2\pi d}\left[\cancel{\vec{x}} + \vec{z} - \cancel{\vec{x}} + \frac{x^2}{d^2}\vec{x}\right]2 = -\frac{\mu_0 I^2}{\pi d}\left[\vec{z} + \frac{x^2}{d^2}\vec{x}\right]$$

Wires are attracting, so a displacement perpendicular to the plane will produce oscillatory motion.
A displacement in the plane is unstable, as the attractive force of the wire in the same direction becomes greater while the attractive force of the opposite wire becomes less

EMI-5



put charge q' at $\frac{R^2}{r}$

$$\frac{-q'}{(R - R^2/r)} = \frac{q}{(r-R)}$$

$$\Rightarrow q' = -q\left(\frac{R - R^2/r}{r-R}\right) = -\left(\frac{R}{r}\right) q$$

$$\frac{R - R^2/r}{r-R} = \frac{R}{r}\left(\frac{r-R}{r-R}\right) = \frac{R}{r}$$

$$\Rightarrow \phi(|\vec{r}'| = R) = \frac{q}{|\vec{r}'-\vec{r}|} - \frac{(R/r)q}{\underbrace{|\vec{r}' - (\frac{R}{r})^2\vec{r}|}_{\left(\frac{R}{r}\right)\left|\frac{r}{R}\vec{r}' - \frac{R}{r}\vec{r}\right| = \left(\frac{R}{r}\right)|\vec{r}'-\vec{r}|}} = 0$$

$$\frac{q}{\cancel{\left(\frac{R}{r}\right)}|\vec{r}'-\vec{r}|}$$

$$\phi(\vec{x}) = \frac{q}{|\vec{x}-\vec{r}|} - \frac{(R/r)q}{|\vec{x} - (\frac{R}{r})^2\vec{r}|} + \frac{Q + (R/r)q}{|\vec{x}|}$$

so that conductor has charge Q

Let charge at $\vec{r}_0 = z_0\hat{z}$

inside sphere: $\Phi_{in} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$ +

outside sphere: $\Phi_{out} = \sum_{l=0}^{\infty} B_l r^{-l-1} P_l(\cos\theta) + \frac{q}{|\vec{r}-\vec{r}_0|}$

but $\frac{1}{|\vec{r}-\vec{r}_0|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\theta) = \sum_{l=0}^{\infty} P_l(\cos\theta)\begin{cases} r^l/r_0^{l+1}, & r < r_0 \\ r_0^l/r^{l+1}, & r > r_0 \end{cases}$

and $\epsilon \frac{\partial \Phi_{in}}{\partial r}\Big|_{r=R} = \frac{\partial \Phi_{out}}{\partial r}\Big|_{r=R}$, $\frac{\partial \Phi_{in}}{\partial \theta}\Big|_{r=R} = \frac{\partial \Phi_{out}}{\partial \theta}\Big|_{r=R}$ $\left(\frac{\partial P_l}{\partial \theta} = P_l'\right)$

$$\Rightarrow \epsilon l A_l R^{l-1} = -(l+1) B_l R^{-l-2} + q l R^{l-1} r_0^{-l-1},$$

$$A_l R^l = B_l R^{-l-1} + q R^l r_0^{-l-1} \Rightarrow B_l R^{-l-2} = q R^{l-1} r_0^{-l-1} - A_l R^{l-1}$$

$$\Rightarrow \epsilon l A_l R^{l-1} = -(l+1) R^{l-1}(q r_0^{-l-1} - A_l) + q l R^{l-1} r_0^{-l-1}$$

$$\Rightarrow A_l(\epsilon l \cancel{R^{l-1}} - (l+1)\cancel{R^{l-1}}) = (-(l+1) + l) q r_0^{-l-1} \cancel{R^{l-1}}$$

$$A_l((\epsilon-1)l - 1) = -q r_0^{-l-1} \Rightarrow A_l = \frac{q}{1-(\epsilon-1)l}\frac{1}{r_0^{l+1}}$$

$$B_l = (q r_0^{-l-1} - A_l) R^{2l+1} = q\left(1 - \frac{1}{1-(\epsilon-1)l}\right)\frac{R^{2l+1}}{r_0^{l+1}} = \frac{-(\epsilon-1) l q R^{2l+1}}{1-(\epsilon-1)l \; r_0^{l+1}}$$

2014

$$\Phi_{in} = q\sum_{l=0}^{\infty}\frac{1}{1-(\epsilon-1)l}\frac{r^l}{r_0^{l+1}}P_l(\cos\theta)$$

$$\Phi_{out} = -q\frac{R}{r_0}\sum_{l=0}^{\infty}\frac{(\epsilon-1)l}{1-(\epsilon-1)l}\left(R^2/r_0\right)^l/r^{l+1}\,P_l(\cos\theta)$$

Does not resemble a charge distribution, except in the case $\epsilon\to\infty$ where

$$\Phi_{out}\to -q\frac{R}{r_0}\left(-\frac{1}{r}+\sum_{l=0}^{\infty}\frac{(R^2/r_0)^l}{r^{l+1}}P_l(\cos\theta)\right)$$

corresponding to image charges $-qR/r_0$ at $z=\frac{R^2}{r_0}$ and qR/r_0 at $z=0$.

QMI-1 $H=\frac{(p-eA)^2}{2m}+e\Phi = \frac{p^2-e(\vec{p}\cdot\vec{A}+\vec{A}\cdot\vec{p})+e^2A^2}{2m}+e\phi$ $(c=1)$

gauge: $\vec{A}\to\vec{A}+\nabla\Lambda$, $\phi\to\phi+\lambda$ (static)

note $\vec{p}=-i\hbar\nabla$ in position basis $\to \vec{p}=-i\nabla$ $(\hbar=1)$

$$H=\frac{(-i\nabla-e\vec{A})^2}{2m}+e\phi \to H'=\frac{(-i\nabla-e\vec{A}-e\nabla\Lambda)^2}{2m}+e\phi+e\lambda$$

$H\psi=i\partial_t\psi \to H'\psi'=i\partial_t\psi'$

$\Rightarrow$ λ introduces factor $e^{ie\lambda t}$ since $i\partial_t(e^{ie\lambda t}\psi)=(-e\lambda+i\partial_t)e^{ie\lambda t}\psi$

Λ introduces factor $e^{ie\Lambda}$ since

$-i\nabla(e^{ie\Lambda}\psi)=(-i\nabla+e\nabla\Lambda)(e^{ie\Lambda}\psi)$

so $\psi\to e^{ie(\lambda t+\Lambda)}\psi$

$A'=A''+\frac{2e_m}{r\sin\theta}\hat{\phi} \Rightarrow \nabla\Lambda=\frac{2e_m}{r\sin\theta}\hat{\phi}\Rightarrow \Lambda=2e_m\phi$

$\psi'=e^{ie2e_m\phi}\psi'' \Rightarrow 2ee_m=N$ for single-valuedness

$\Rightarrow e_m=\frac{N}{2e}$, $N\in\mathbb{Z}$

QM1-2 $H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{k}{2}x_1^2 + \frac{k}{2}x_2^2 + c\, x_1 x_2$, $x_+ = \frac{1}{\sqrt{2}}(x_1 + x_2)$, $x_- = \frac{1}{\sqrt{2}}(x_1 - x_2)$

$\Rightarrow x_+^2 + x_-^2 = x_1^2 + x_2^2$

$x_+^2 - x_-^2 = 2x_1 x_2$

$p_\pm = \frac{1}{\sqrt{2}}(p_1 \pm p_2)$

$$H = \frac{p_+^2}{2m} + \frac{p_-^2}{2m} + \frac{k}{2}x_+^2 + \frac{k}{2}x_-^2 + \frac{c}{2}(x_+^2 - x_-^2)$$

$$= \frac{p_+^2}{2m} + \frac{p_-^2}{2m} + \frac{k+c}{2}x_+^2 + \frac{k-c}{2}x_-^2$$

$[x_\pm, p_\pm] = \frac{1}{2}([x_1,p_1] + [x_2,p_2]) = i$

$[x_\pm, p_\mp] = \frac{1}{2}([x_1,p_1] - [x_2,p_2]) = 0$

$\Rightarrow \omega_+ = \sqrt{\frac{k+c}{m}}$, $\omega_- = \sqrt{\frac{k-c}{m}}$

uncoupled oscillators with energy $(n_+ + \frac{1}{2})\omega_+$, $(n_- + \frac{1}{2})\omega_-$

$\Rightarrow$ total energy $E_{n_+ n_-} = (n_+ + \frac{1}{2})\sqrt{\frac{k+c}{m}} + (n_- + \frac{1}{2})\sqrt{\frac{k-c}{m}}$, $n_\pm = 0, 1, 2, \ldots$

QM1-3 $j_1 = j_2 = \frac{1}{2}$. $|1\,1\rangle = |\tfrac{1}{2}\,\tfrac{1}{2}\rangle$, $|1\,0\rangle = \frac{1}{\sqrt{2}}\left(|\tfrac{1}{2}\,\tfrac{-1}{2}\rangle + |\tfrac{-1}{2}\,\tfrac{1}{2}\rangle\right)$,

$|1\,{-1}\rangle = |\tfrac{-1}{2}, \tfrac{-1}{2}\rangle$, $|0\,0\rangle = \frac{1}{\sqrt{2}}\left(|\tfrac{1}{2}\,\tfrac{-1}{2}\rangle - |\tfrac{-1}{2}\,\tfrac{1}{2}\rangle\right)$

$$\underbrace{D^{(\frac{1}{2})}}_{\text{dimension 2}} \otimes D^{(\frac{1}{2})} = \underbrace{D^{(0)}}_{\text{dimension 1}} \oplus \underbrace{D^{(1)}}_{\text{dimension 3}}$$

note $J_\pm |j\,m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\, |j, m \pm 1\rangle$

$J_-|1\,1\rangle = \sqrt{2}\,|1\,0\rangle = |\tfrac{-1}{2}\,\tfrac{1}{2}\rangle + |\tfrac{1}{2}\,\tfrac{-1}{2}\rangle$ $\quad (|1\,1\rangle = |\tfrac{1}{2}\,\tfrac{1}{2}\rangle)$

$\Rightarrow |1\,0\rangle = \frac{1}{\sqrt{2}}\left(|\tfrac{-1}{2}\,\tfrac{1}{2}\rangle + |\tfrac{1}{2}\,\tfrac{-1}{2}\rangle\right)$

$J_-|1\,0\rangle = \sqrt{2}\,|1\,{-1}\rangle = \frac{1}{\sqrt{2}}\left(|\tfrac{-1}{2}\,\tfrac{-1}{2}\rangle + |\tfrac{-1}{2}\,\tfrac{-1}{2}\rangle\right)$

$\Rightarrow |1\,{-1}\rangle = |\tfrac{-1}{2}\,\tfrac{-1}{2}\rangle$

$|\tfrac{1}{2}\,\tfrac{-1}{2}\rangle = \frac{1}{\sqrt{2}}|1\,0\rangle + \frac{1}{\sqrt{2}}|0\,0\rangle$

$|\tfrac{-1}{2}, \tfrac{1}{2}\rangle = \frac{1}{\sqrt{2}}|1\,0\rangle - \frac{1}{\sqrt{2}}|0\,0\rangle$

sign ambiguity, but must be opposite

$\Rightarrow |0\,0\rangle = \frac{1}{\sqrt{2}}\left(|\tfrac{1}{2}\,\tfrac{-1}{2}\rangle - |\tfrac{-1}{2}\,\tfrac{1}{2}\rangle\right)$

2014 QM1-4 $\langle\psi|e^{iH\Delta t}|\psi\rangle = \langle\psi|1 - iH\Delta t - H^2\Delta t^2/2 + \dots|\psi\rangle$

$= 1 - i\bar{H}\Delta t - \langle\psi|H^2|\psi\rangle \Delta t^2/2 + \dots$ $(\hbar = 1)$

$|\langle\psi|e^{iH\Delta t}|\psi\rangle|^2 = (1 - \tfrac{1}{2}\langle\psi|H^2|\psi\rangle\Delta t^2)^2 + \bar{H}^2\Delta t^2 + \dots$

$= 1 - \langle\psi|H^2|\psi\rangle\Delta t^2 + \bar{H}^2\Delta t^2$

$= 1 - \Delta t^2(2\Delta t^2$

note $\Delta E^2 = \langle\psi|(H-\bar{H})^2|\psi\rangle = \langle\psi|H^2|\psi\rangle \underbrace{- 2\langle\psi|H\bar{H}|\psi\rangle}_{-2\bar{H}^2} + \underbrace{\langle\psi|\bar{H}^2|\psi\rangle}_{\bar{H}^2}$

$= \langle\psi|H^2|\psi\rangle - \bar{H}^2$

$|\langle\psi|e^{-iH\Delta t}|\psi\rangle|^2 = 1 - \frac{\Delta E^2\Delta t^2}{\hbar^2}$

Now $H = a\sigma_z$, $|\psi\rangle = |S_x+\rangle$ $H = a\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

$S_x|\psi\rangle = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{pmatrix} \Rightarrow |\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}$

$e^{-iH\Delta t/\hbar} = \begin{pmatrix}e^{-i\frac{a}{\hbar}\Delta t} & 0\\ 0 & e^{i\frac{a}{\hbar}\Delta t}\end{pmatrix}$

$|\langle\psi|e^{-iH\Delta t/\hbar}|\psi\rangle|^2 = \left|\frac{1}{2}(1\ 1)\begin{pmatrix}e^{-i\frac{a}{\hbar}\Delta t} & 0\\ 0 & e^{i\frac{a}{\hbar}\Delta t}\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix}\right|^2$

$= \frac{1}{4}\left|(1\ 1)\begin{pmatrix}e^{-i\frac{a}{\hbar}\Delta t}\\ e^{i\frac{a}{\hbar}\Delta t}\end{pmatrix}\right|^2 = \frac{1}{4}\left|2\cos(\tfrac{a}{\hbar}\Delta t)\right|^2$

$= \cos^2(\tfrac{a}{\hbar}\Delta t)$

$\simeq \left(1 - \tfrac{1}{2}(\tfrac{a}{\hbar}\Delta t)^2\right)^2 \simeq 1 - (\tfrac{a}{\hbar}\Delta t)^2$

$\langle\psi|H|\psi\rangle = a\frac{1}{2}(1\ 1)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} = \frac{a}{2}(1\ 1)\begin{pmatrix}1\\ -1\end{pmatrix} = 0$

$\langle\psi|H^2|\psi\rangle = \frac{a^2}{2}(1\ 1)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} = \frac{a^2}{2}(1\ 1)\begin{pmatrix}1\\ 1\end{pmatrix} = a^2$

$\Rightarrow \Delta E^2 = a^2$

$\Rightarrow |\langle\psi|e^{-iH\Delta t/\hbar}|\psi\rangle|^2 \simeq 1 - \frac{\Delta E^2\Delta t^2}{\hbar^2}$

QM1-5 $j_1 = 1,\ j_2 = \frac{1}{2} \Rightarrow j = \frac{3}{2}$ or $\frac{1}{2}$ $\left(J^2 = \frac{3}{4}\hbar^2 \text{ or } \frac{15}{4}\hbar^2\right)$

$\underbrace{j = \frac{1}{2},\ m = \pm\frac{1}{2}}_{2} \; ; \; + \; \underbrace{j = \frac{3}{2},\ m = \pm\frac{3}{2}, \pm\frac{1}{2}}_{4} = 6$ states

$H = \frac{a}{\hbar^2}\vec{S}_1 \cdot \vec{S}_2 + \frac{b}{\hbar} S_z$, $\quad S_z = S_{1z} + S_{2z}$

$\vec{S} = \vec{S}_1 + \vec{S}_2 \quad \vec{S}^2 = \vec{S}_1^2 + \vec{S}_2^2 + 2\vec{S}_1 \cdot \vec{S}_2$

$\Rightarrow \vec{S}_1 \cdot \vec{S}_2 = \frac{1}{2}\left(\vec{S}^2 - \vec{S}_1^2 - \vec{S}_2^2\right)$

$= \frac{1}{2}\left(\vec{S}^2 - 2\hbar^2 - \frac{3}{4}\hbar^2\right)$

$= \frac{1}{2}\vec{S}^2 - \frac{11}{8}\hbar^2$

$H = \frac{a}{2\hbar^2}\vec{S}^2 - \frac{11}{8}a + \frac{b}{\hbar}S_z \quad \Rightarrow H|j,m\rangle = \left(\frac{a}{2}j(j+1) - \frac{11}{8}a + bm\right)|j,m\rangle$

$j = \frac{1}{2},\ m = \pm\frac{1}{2} \quad ; \quad j = \frac{3}{2},\ m = \pm\frac{3}{2}, \pm\frac{1}{2}$

$\vec{S}^2 = \frac{3}{4}\hbar^2 \quad S_z = \pm\frac{\hbar}{2} \qquad \vec{S}^2 = \frac{15}{4}\hbar^2 \quad S_z = \pm\frac{3\hbar}{2}, \pm\frac{\hbar}{2}$

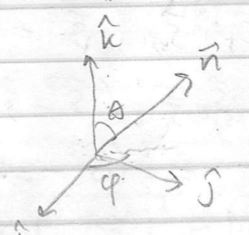
$E = \frac{3}{8}a - \frac{11}{8}a \pm \frac{b}{2} \qquad E = \frac{15}{8}a - \frac{11}{8}a \pm \frac{3b}{2}$ or $\frac{b}{2}$

$= -a \pm \frac{b}{2} \qquad = \frac{1}{2}a \pm \frac{3b}{2}, \frac{b}{2}$

$\Rightarrow E = -a \pm \frac{b}{2}\ ,\ \frac{a}{2} \pm \frac{3b}{2}\ ,\ \frac{a}{2} \pm \frac{b}{2}$

2014 EMII-1 incident light unpolarized ⇒ need to average $\vec{d}$ over xy-plane. $\vec{d} = d(\hat{i}\cos\alpha + \hat{j}\sin\alpha)$

$$\frac{dP}{d\Omega} = \frac{1}{2\pi}\int_0^{2\pi} d\alpha \, \frac{c}{8\pi} k^4 d^2 \underbrace{\left|\left(\vec{n}\times(\hat{i}\cos\alpha + \hat{j}\sin\alpha)\right)\times\vec{n}\right|^2}_{\left|(\vec{n}\times\hat{i})\times\vec{n}\cos\alpha + (\vec{n}\times\hat{j})\times\vec{n}\sin\alpha\right|^2}$$



note $(\vec{n}\times\hat{i})\times\vec{n}\cdot(\vec{n}\times\hat{j})\times\vec{n} =$
$[\vec{n}(\vec{n}\cdot\hat{i}) - \hat{i}]\cdot[\vec{n}(\vec{n}\cdot\hat{j}) - \hat{j}] = -(\vec{n}\cdot\hat{i})(\vec{n}\cdot\hat{j})$

$$\frac{dP}{d\Omega} = \frac{1}{2\pi}\int_0^{2\pi} d\alpha \, \frac{c}{8\pi} k^4 d^2 \left(\underbrace{|(\vec{n}\times\hat{i})\times\vec{n}|^2}_{} \underbrace{\cos^2\alpha}_{\pi} + |(\vec{n}\times\hat{j})\times\vec{n}|^2 \underbrace{\sin^2\alpha}_{\pi} - 2(\vec{n}\cdot\hat{i})(\vec{n}\cdot\hat{j}) \underbrace{\sin\alpha\cos\alpha}_{0} \right)$$

$$|-\vec{n}(\vec{n}\cdot\hat{i}) + \hat{i}|^2 = (\vec{n}\cdot\hat{i})^2 - 2(\vec{n}\cdot\hat{i})^2 + 1 = 1 - (\vec{n}\cdot\hat{i})^2$$

$$= \frac{1}{2}\frac{c}{8\pi} k^4 d^2 \Big(2 - \underbrace{(\vec{n}\cdot\hat{i})}_{\sin\theta\cos\varphi}{}^2 - \underbrace{(\vec{n}\cdot\hat{j})}_{\sin\theta\sin\varphi}{}^2 \Big)$$

$$= \frac{c}{16\pi} k^4 d^2 (2 - \sin^2\theta) = \frac{c}{16\pi} k^4 d^2 (1 + \cos^2\theta)$$

$\vec{d}$ is in xy-plane

For $\theta \sim \pi/2$, this means scattered $\vec{E}$ is polarized parallel to xy-plane.
For $\theta \sim 0$, scattered $\vec{E}$ is unpolarized.
Between these extremes, there is partial polarization

EMII-2 cm frame: $2E_e' = 2(m_e + m_\mu)$, $E_e = \gamma' m_e$ (since we boost by γ' to get back to lab)

$$\Rightarrow \gamma' = \frac{m_e + m_\mu}{m_e} = 1 + \frac{m_\mu}{m_e}$$

boost to lab frame: $\gamma' = (1 - v'^2)^{-1/2} \Rightarrow v' = (1 - \gamma'^{-2})^{1/2}$

$$\begin{pmatrix} \gamma' & \gamma'(1-\gamma'^{-2})^{1/2} \\ \gamma'(1-\gamma'^{-2})^{1/2} & \gamma' \end{pmatrix} \begin{pmatrix} \gamma' m_e \\ \pm(\gamma'^2 - 1)^{1/2} m_e \end{pmatrix} = \begin{pmatrix} \gamma'^2 m_e \pm \gamma'^2 (1-\gamma'^{-2}) m_e \\ \gamma'^2 m_e \left((1-\gamma'^{-2})^{1/2} \pm (1-\gamma'^{-2})^{1/2} \right) \end{pmatrix}$$

lab frame: $E_e = m_e, \gamma m_e$

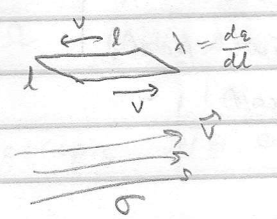
$$= \gamma'^2 m_e (1 \pm (1 - \gamma'^{-2})) = m_e(\gamma'^2 \pm (\gamma'^2 - 1))$$

$$= m_e,\ (2\gamma'^2 - 1) m_e$$

$$\Rightarrow \gamma = 2\gamma'^2 - 1 = 2\left(1 + \frac{m_\mu}{m_e}\right)^2 - 1$$

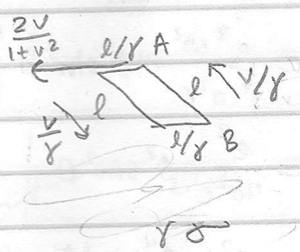
$m_\mu = 105.7$ MeV, $m_e = 0.5110$ MeV $\rightarrow \gamma' = 207.8$, $\gamma = 86402$

EMII-3



$\lambda = \frac{dq}{dl}$ boost to S'

$j = \lambda v$



$$\frac{dx'}{dt'} = \frac{\gamma v\,dt + \gamma\,dx}{\gamma\,dt + \gamma v\,dx} = \frac{v + dx/dt}{1 + v\frac{dx}{dt}} \qquad \begin{pmatrix} dt \\ dx \\ dy \\ dz \end{pmatrix}' = \begin{pmatrix} \gamma & \gamma v & 0 & 0 \\ \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dt \\ dx \\ dy \\ dz \end{pmatrix}$$

$$\frac{dy'}{dt'} = \frac{dy}{\gamma\,dt + \gamma v\,dx} = \frac{1}{\gamma} \frac{dy/dt}{1 + v\,dx/dt}$$

$$A: \begin{pmatrix} \lambda' \\ j' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma v \\ \gamma v & \gamma \end{pmatrix} \begin{pmatrix} \lambda \\ -\lambda v \end{pmatrix} = \begin{pmatrix} \gamma\lambda + \gamma\lambda v^2 \\ -2\gamma\lambda v \end{pmatrix} = \begin{pmatrix} \gamma\lambda(1+v^2) \\ -2\gamma\lambda v \end{pmatrix} \qquad q_A = \gamma\lambda l(1+v^2)$$

$$B: \begin{pmatrix} \lambda' \\ j' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma v \\ -\gamma v & \gamma \end{pmatrix} \begin{pmatrix} \lambda \\ \lambda v \end{pmatrix} = \begin{pmatrix} \gamma\lambda - \gamma\lambda v^2 \\ -\gamma\lambda v + \gamma\lambda v \end{pmatrix} = \begin{pmatrix} \gamma\lambda(1-v^2) \\ 0 \end{pmatrix} \qquad q_B = \gamma\lambda l(1-v^2)$$

$$\vec{d} = \frac{q_A - q_B}{2} l\,\hat{y} = \gamma\lambda l v^2 l\,\hat{y} = \gamma\lambda l^2 v^2\,\hat{y}$$

$$\vec{\tau} = \vec{d} \times \vec{E}, \qquad \vec{E} = 4\pi \frac{\gamma\sigma}{2}\hat{z} = 2\pi\gamma\sigma\,\hat{z}$$

$$\vec{\tau} = 2\pi\gamma^2\sigma\lambda l^2 v^2\,\hat{x}$$

2014 EMII-4

$\vec{n} = (\sin\theta, 0, \cos\theta)$, $\vec{\beta}_1 = (0, 0, \beta)$, $\vec{x}_1 = (0, 0, c\beta t')$

$\vec{n}\times\vec{\beta}_1 = (0, -\beta\sin\theta, 0) \Rightarrow \vec{n}\times(\vec{n}\times\vec{\beta}_1) = (\beta\sin\theta\cos\theta, 0, -\beta\sin^2\theta)$

$\vec{n}\cdot\vec{x}_1 = c\beta t'\cos\theta$

$$\left.\frac{dW}{d\omega d\Omega}\right|_1 = \frac{e^2\omega^2}{4\pi^2 c}\left|\underbrace{\int_0^\infty dt'\,\beta\sin\theta(\hat{i}\cos\theta - \hat{k}\sin\theta)e^{i\omega t'(1-\beta\cos\theta)}}_{\beta\sin\theta(\hat{i}\cos\theta - \hat{k}\sin\theta)\frac{-1}{i\omega(1-\beta\cos\theta)}}\right|^2$$

$$= \frac{e^2\omega^2}{4\pi^2 c}\frac{\beta^2\sin^2\theta}{\omega^2(1-\beta\cos\theta)^2} = \frac{e^2}{4\pi^2 c}\frac{\beta^2\sin^2\theta}{(1-\beta\cos\theta)^2} \quad \text{(one electron)}$$

$2e^-$:

$$\frac{dW}{d\omega d\Omega} = \frac{e^2\omega^2}{4\pi^2 c}\left|\int_0^\infty dt'\left\{\vec{n}\times(\vec{n}\times\vec{\beta})e^{i\omega(t'-\vec{n}\cdot\vec{x}(t')/c)} - \vec{n}\times(\vec{n}\times\vec{\beta})e^{i\omega(t'+\vec{n}\cdot\vec{x}(t')/c)}\right\}\right|^2$$

$$= \frac{e^2\omega^2}{4\pi^2 c}\left|\int_0^\infty dt'\,\beta\sin\theta(\hat{i}\cos\theta - \hat{k}\sin\theta)e^{i\omega t'}\left\{e^{-i\omega\beta t'\cos\theta} - e^{i\omega\beta t'\cos\theta}\right\}\right|^2$$

$$= \frac{e^2\omega^2}{4\pi^2 c}\cdot 4\beta^2\sin^2\theta\left|\frac{-1}{i\omega(1-\beta\cos\theta)} - \frac{-1}{i\omega(1+\beta\cos\theta)}\right|^2$$

$$= \frac{e^2}{4\pi^2 c}\beta^2\sin^2\theta\left|\frac{\cancel{1}+\beta\cos\theta - (\cancel{1}-\beta\cos\theta)}{1-\beta^2\cos^2\theta}\right|^2$$

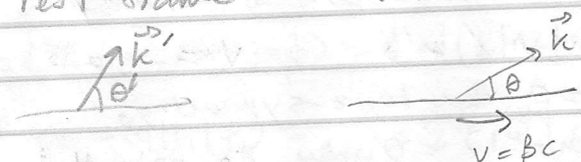
$$= \frac{e^2}{\pi^2 c}\frac{\beta^4\sin^2\theta\cos^2\theta}{(1-\beta^2\cos^2\theta)^2}$$

at $\theta = \pm\pi/2$ the radiation from $2e^-$ interferes.

nonrel:

$$\frac{dW}{d\omega d\Omega} \simeq \frac{e^2}{\pi^2 c}\beta^4\sin^2\theta\cos^2\theta \sim |Y_{l=2}|^2 \quad \text{quadrupole radiation}$$

EMII-8 rest frame lab frame



$$\begin{pmatrix} k' \\ k_x' \\ k_y' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 \\ -\gamma\beta & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} k \\ k_x \\ k_y \end{pmatrix} = \begin{pmatrix} \gamma k - \gamma\beta k_x \\ -\gamma\beta k + \gamma k_x \\ k_y' \end{pmatrix}$$

$$\omega' = k' = \sqrt{k_x'^2 + k_y'^2}$$

$$\cos\theta' = \frac{k_x'}{k'} = \frac{-\gamma\beta k + \gamma k_x}{\gamma k - \gamma\beta k_x} = \frac{k_x/k - \beta}{1 - \beta k_x/k} = \frac{\cos\theta - \beta}{1 - \beta\cos\theta}$$

$$d\Omega = \sin\theta\, d\theta\, d\varphi$$

$$\frac{d\Omega'}{d\Omega} = \frac{\sin\theta' d\theta'}{\sin\theta\, d\theta} = \frac{d\cos\theta'}{d\cos\theta} = \frac{1 - \cancel{\beta\cos\theta} + \beta(\cancel{\cos\theta} - \beta)}{(1-\beta\cos\theta)^2} = \frac{1}{\gamma^2(1-\beta\cos\theta)^2}$$

$$\frac{dN}{d\Omega} = \frac{dN}{d\Omega'}\frac{d\Omega'}{d\Omega} = \frac{f(\theta', \phi')}{\gamma^2(1-\beta\cos\theta)^2}$$

isotropic: $f(\theta', \phi') = f$ constant

$$\left.\frac{dN}{d\Omega}\right|_{\theta=0} = \frac{f}{\gamma^2(1-\beta)^2} \qquad \left.\frac{dN}{d\Omega}\right|_{\theta=\pi} = \frac{f}{\gamma^2(1+\beta)^2}$$

ratio $\left(\frac{1+\beta}{1-\beta}\right)^2$, $\gamma^{-2} = 1 - \beta^2 \rightarrow \beta = (1-\gamma^{-2})^{1/2}$

$$\simeq 1 - \tfrac{1}{2}\gamma^{-2}$$

$$\simeq \left(\frac{2}{\frac{1}{2}\gamma^{-2}}\right)^2 = 16\gamma^4$$

2014 QM II-1 $n=2 \rightarrow l=0, m=0; \ l=1, m=\pm1, 0$ (4 states)

matrix elements $\langle l,m|V|l',m'\rangle$, $V=-ez|\vec{E}|=-e|\vec{E}|r\cos\theta$

note $\langle l,m|V|l,m\rangle=0$ due to z-symmetry;

$\underbrace{\langle l,m\neq\pm1|V|1,m=\pm1\rangle}$ $=0$ due to azimuthal symmetry of

only $\langle 00|V|10\rangle$ (and $\langle 10|V|00\rangle$) nonzero

Let $a=\langle 00|V|10\rangle$

$$V=\begin{array}{c|cccc} & 00 & 10 & 1\text{-}1 & 11 \\ 00 & & & & \\ 10 & & & & \\ 1\text{-}1 & & & & \\ 11 & & & & \end{array}\quad V=\begin{pmatrix} 0 & a^* & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \lambda^4-\lambda^2|a|^2=0$$

$$\Rightarrow \lambda=0, \ \lambda=\pm|a|=\pm a \quad (a=a^*)$$

eigenvectors $\begin{pmatrix} 0 & a \\ a & 0\end{pmatrix}\begin{pmatrix} v_1 \\ v_2\end{pmatrix}=\pm a\begin{pmatrix} v_1 \\ v_2\end{pmatrix} \Rightarrow \begin{matrix} av_2=\pm av_1 \\ av_1=\pm av_2\end{matrix}$

$\Rightarrow \frac{1}{\sqrt{2}}\left(|00\rangle\pm|10\rangle\right)$

$$a=\left(\frac{z}{2a_0}\right)^{3}\frac{\sqrt{3}}{4\pi}(-e|\vec{E}|)\int r^2dr\left(2-\frac{zr}{a_0}\right)\frac{zr}{\sqrt{3}a_0}e^{-zr/a_0}$$
$$\cdot\int\sin\theta d\theta\cos\theta\cdot 2\pi\cdot r\cos\theta$$

$$=\frac{-e|\vec{E}|}{a_0^4}\cdot 2\int_0^\infty r^4\left(1-\frac{r}{a_0}\right)e^{-2r/a_0}dr\underbrace{\int_{-1}^{1}\mu^2d\mu}_{2/3}$$

$$-\int\frac{r^5}{a_0}e^{-2r/a_0}dr=-\int\frac{5r^4}{\cancel{a_0}}\frac{\cancel{a_0}}{2}e^{-2r/a_0}dr$$

$$=\frac{e|\vec{E}|}{a_0^4}\cdot 2\int_0^\infty r^4e^{-2r/a_0}dr=\frac{e|\vec{E}|}{a_0^3}\cdot 4\int_0^\infty r^3e^{-2r/a_0}=\frac{6e|\vec{E}|}{a_0^2}\int_0^\infty r^2e^{-2r/a_0}dr$$

$$=\frac{6e|\vec{E}|}{a_0}\int_0^\infty re^{-2r/a_0}dr=3e|\vec{E}|\int_0^\infty e^{-2r/a_0}dr=\frac{3}{2}e|\vec{E}|a_0$$

$\Rightarrow \Delta E=\pm\frac{3}{2}e|\vec{E}|a_0$ corresp. $|\pm\rangle=\frac{1}{\sqrt{2}}\left(|00\rangle\pm|10\rangle\right)$

QM11-2 $\vec{J}^2 = (\vec{L}+\vec{S})^2 = \vec{L}^2 + \vec{S}^2 + 2\vec{L}\cdot\vec{S} \Rightarrow \vec{L}\cdot\vec{S} = \frac{1}{2}(\vec{J}^2 - \vec{L}^2 - \vec{S}^2)$

$\Rightarrow \vec{L}\cdot\vec{S}\,|n=2, l=1, j=\frac{3}{2}\rangle = \frac{1}{2}\hbar^2\left(\frac{15}{4} - 2 - \frac{3}{4}\right) = \frac{1}{2}\hbar^2$

so $\Delta E = \langle\psi|H|\psi\rangle = \frac{1}{2m_e^2c^2}\frac{1}{2}\hbar^2 \underbrace{\langle\psi|\frac{1}{r}\frac{dV_c}{dr}|\psi\rangle}_{I_r} = \frac{\hbar^2}{4m_e^2c^2} I_r$

$$I_r = \left(\frac{1}{2a_0}\right)^3 \frac{1}{3a_0^2}\int_0^\infty r^2 dr\, r^2 e^{-r/a_0}\frac{1}{r}\frac{dV_c}{dr}$$

$$= \frac{1}{24a_0^5}\int_0^\infty r^3 e^{-r/a_0} dV_c = \frac{1}{24a_0^5}\left(\frac{-V_0}{\mu}\right)\int_0^\infty \left(3r^2 - \frac{r^3}{a_0}\right)e^{-r/a_0}\frac{e^{-\mu r}}{r}\,dr$$

$$= \frac{V_0}{24\mu a_0^5}\int_0^\infty \left(\frac{r^2}{a_0} - 3r\right)e^{-(\frac{1}{a_0}+\mu)r}dr$$

$$\int \frac{r^2}{a_0}e^{-(\frac{1}{a_0}+\mu)r}dr = \int \frac{2r}{a_0}\frac{1}{(\frac{1}{a_0}+\mu)}e^{-\cdots}$$

$$= \int \frac{2r\,dr}{1+\mu a_0}e^{-(\frac{1}{a_0}+\mu)r}$$

$$= \frac{V_0}{24\mu a_0^5}\left(\frac{2}{1+\mu a_0} - 3\right)\underbrace{\int_0^\infty r e^{-(\frac{1}{a_0}+\mu)r}dr}_{\frac{1}{\frac{1}{a_0}+\mu}\int_0^\infty e^{-(\frac{1}{a_0}+\mu)r}dr = \frac{1}{(\frac{1}{a_0}+\mu)^2} = \frac{a_0^2}{(1+\mu a_0)^2}}$$

$$= \frac{V_0}{24\mu a_0^3}\left(\frac{2}{1+\mu a_0} - 3\right)\frac{1}{(1+\mu a_0)^2}$$

$$\simeq -\frac{V_0}{24\mu a_0^3} \Rightarrow \Delta E \simeq \frac{-\hbar^2}{96m_e^2c^2a_0^3}\frac{V_0}{\mu}, \quad \frac{V_0}{\mu} \simeq ke^2$$

$$\Rightarrow \Delta E \simeq \frac{-\hbar^2 ke^2}{96m_e^2c^2a_0^3}$$

2014 QMII-3 unperturbed eigenstates $\binom{1}{0}$ with energy E_0 and $\binom{0}{1}$ with energy $-E_0$

$$\Rightarrow \omega_{12} = \frac{2E_0}{\hbar} \qquad V_{12} = E_1 e^{i\omega t}, \quad V_{21} = E_1 e^{-i\omega t}$$

$$i\hbar \frac{d}{dt} c_2(t) = E_1 e^{i(-\omega+\omega_{12})t} c_1(t) \simeq E_1 c_1(t)$$

$$i\hbar \frac{d}{dt} c_1(t) = E_1 e^{i(\omega-\omega_{12})t} c_2(t) \simeq E_1 c_2(t)$$

$$\Rightarrow \ddot{c}_2 = \frac{E_1}{i\hbar} \dot{c}_1 = -\left(\frac{E_1}{\hbar}\right)^2 c_2$$

$$\Rightarrow c_2(t) = A \sin\left(\frac{E_1}{\hbar} t\right), \quad c_1(t) = A \cos\left(\frac{E_1}{\hbar} t\right)$$

$$|c_1|^2 + |c_2|^2 = A^2 = 1 \Rightarrow A = 1$$

$$|c_2(t)|^2 = \sin^2\left(\frac{E_1}{\hbar} t\right)$$

$$|c_1|^2 = |c_2|^2 \Rightarrow \sin^2\left(\frac{E_1}{\hbar} t\right) = \cos^2\left(\frac{E_1}{\hbar} t\right) = \frac{1}{2}$$

$$\Rightarrow \frac{E_1}{\hbar} t = \frac{\pi}{4} \Rightarrow t = \frac{\pi}{4} \frac{\hbar}{E_1}$$

semiclassically, pointed in x direction $(c_1 = c_2)$

QMII-4 $V_0 = -\alpha \delta(x) \Rightarrow -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi - \alpha \delta(x) \psi = E_0 \psi$

$$\psi\big|_{x<0} = A e^{\kappa x}, \quad \psi\big|_{x>0} = A e^{-\kappa x} \qquad -\frac{\hbar^2 \kappa^2}{2m} = E_0$$

$$\int_{-\infty}^{\infty} \psi^2 dx = 2\int_0^\infty A^2 e^{-2\kappa x} dx = 2A^2 \frac{1}{2\kappa} = \frac{A^2}{\kappa} = 1 \Rightarrow A = \sqrt{\kappa}$$

$$-\frac{\hbar^2}{2m}\left(\psi'\big|_{x>0} - \psi'\big|_{x<0}\right) - \alpha \psi(0) = 0$$

$$-\frac{\hbar^2}{2m}(-\kappa - \kappa) A - \alpha A = 0 \Rightarrow$$

$$\Rightarrow \frac{\hbar^2}{m} \kappa = \alpha \Rightarrow \kappa = \frac{m\alpha}{\hbar^2} \Rightarrow E_0 = -\frac{\hbar^2}{2m} \frac{m^2 \alpha^2}{\hbar^4}$$

$$E_0 = -\frac{\hbar^2 \kappa^2}{2m}, \quad \langle x | \psi_0 \rangle = \sqrt{\kappa} e^{-\kappa |x|}, \quad \kappa = \frac{m\alpha}{\hbar^2}$$

Let $|j\rangle$ be solution to δ at $x = ja$ (energy E_0)

$$|\theta\rangle = \sum_{j=0}^{N-1} e^{ij\theta} |j\rangle \quad \text{for translation symmetry}$$

$$T(a)|\theta\rangle \doteq e^{i\theta}|\theta\rangle, \quad T(Na) = |\theta\rangle = e^{iN\theta}|\theta\rangle \Rightarrow \theta = \frac{2\pi}{N}\cdot n, \; n\in\mathbb{Z}$$

Assume $H|j\rangle = E_0|j\rangle + \Delta|j+1\rangle + \Delta|j-1\rangle$, i.e. $a\varkappa >> 1$

$$\Delta = \langle j+1|H|j\rangle = E_0\langle j+1|j\rangle$$

$$= E_0\varkappa\int dx\, e^{-\varkappa|x|}e^{-\varkappa|x-a|} = E_0\varkappa\left[\int_{-\infty}^{0} e^{\varkappa(2x-a)}dx + \int_0^a e^{-\varkappa a}dx + \int_a^\infty e^{-\varkappa(2x-a)}dx\right]$$

$$= E_0\varkappa\left[e^{-\varkappa a}\frac{1}{2\varkappa} + ae^{-\varkappa a} + e^{\varkappa a}\frac{e^{-2\varkappa a}}{2\varkappa}\right] = E_0 e^{-\varkappa a}(1+\varkappa a)$$

$$H|\theta\rangle = \sum_{j=0}^{N-1} e^{ij\theta} H|j\rangle = \sum_{j=0}^{N-1} e^{ij\theta}\left(\Delta(e^{i\theta}+e^{-i\theta}) + E_0\right)|j\rangle = (E_0 + 2\Delta\cos\theta)|\theta\rangle$$

$$\Rightarrow E_\theta = E_0\left[1+2e^{-\varkappa a}(1+\varkappa a)\cos\theta\right], \quad \theta = \frac{2\pi}{N}\cdot n, \quad n = 1,2,3,\ldots$$

$$\text{with } E_0 = -\frac{\hbar^2\varkappa^2}{2m}, \quad \varkappa = \frac{m\alpha}{\hbar^2}$$

Corresp. $|\theta\rangle = \sum_{j=0}^{N-1} e^{ij\theta}|j\rangle$, $\langle x|j\rangle = \sqrt{\varkappa}\, e^{-\varkappa|x-ja|}$

2014 QMII-5

$$f(\vec{k}',\vec{k}) = -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int d^3x'\, e^{i(\vec{k}-\vec{k}')\cdot\vec{x}'} V(\vec{x}') \quad , \quad V(\vec{x}') = g\,\delta^3(\vec{x}')$$

$$= -\frac{1}{4\pi}\frac{2m}{\hbar^2} g \Rightarrow \frac{d\sigma}{d\Omega} = \left(\frac{mg}{2\pi\hbar^2}\right)^2 \Rightarrow \sigma = \frac{1}{\pi}\frac{m^2g^2}{\hbar^4}$$

$ak \ll 1$: scattering in phase so $\sigma = \frac{4}{\pi}\frac{m^2g^2}{\hbar^4}$ $(\propto f^2)$

$ak \gg 1$: scattering phases independent so $\sigma = \frac{2}{\pi}\frac{m^2g^2}{\hbar^4}$

$$f(\vec{k}',\vec{k}) = -\frac{1}{4\pi}\frac{2m}{\hbar^2} g\left[e^{i(\vec{k}-\vec{k}')\cdot a\hat{z}} + e^{-i(\vec{k}-\vec{k}')\cdot a\hat{z}}\right]$$

$$= -\frac{1}{4\pi}\frac{2m}{\hbar^2} g\, 2\cos\left[a(k_z - k_z')\right] \quad , \quad k_z' = k'\cos\theta = k\cos\theta$$

(if energy conserved)

$$|f(\vec{k}',\vec{k})|^2 = \left(\frac{mg}{\pi\hbar^2}\right)^2 \cos^2\left[ak(\cos\alpha - \cos\theta)\right] = \frac{d\sigma}{d\Omega}$$

$\cos\alpha = k_z/k$

$$\sigma = \int_{-1}^{1} 2\pi\, d\mu \left(\frac{mg}{\pi\hbar^2}\right)^2 \cos^2\left[ak(\cos\alpha - \mu)\right] \quad , \quad \mu = \cos\theta$$

$$= \frac{2}{\pi}\frac{m^2g^2}{\hbar^4}\left[\frac{\sin[2ak(\cos\alpha - \mu)]}{-4ak} + \frac{\mu}{2}\right]_{-1}^{1}$$

$$= \frac{2}{\pi}\frac{m^2g^2}{\hbar^4}\left[1 + \frac{\sin[2ak(1+\cos\alpha)] + \sin[2ak(1-\cos\alpha)]}{4ak}\right]$$

(α = incident polar angle)

$$\sigma\Big|_{ak\ll 1} \to \frac{2}{\pi}\frac{m^2g^2}{\hbar^4}\left[1 + \frac{4ak}{4ak}\right] = \frac{4}{\pi}\frac{m^2g^2}{\hbar^4}$$

$$\sigma\Big|_{ak\gg 1} \to \frac{2}{\pi}\frac{m^2g^2}{\hbar^4}$$