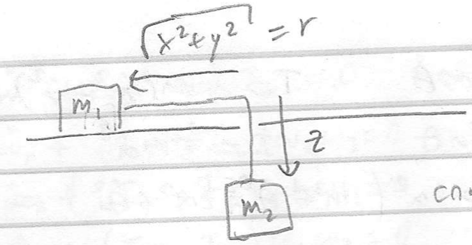
2013 CM1



$T = \frac{1}{2} m_1 (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} m_2 \dot{z}^2$

$V = -m_2 g z$

constraint: $x^2+y^2 = (l-z)^2$ or $r+z=l$

$L = \frac{1}{2} m_1 (\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2} m_2 \dot{z}^2 + m_2 g z = \frac{1}{2} m_1 (\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2} m_2 \dot{r}^2 - m_2 g r$ (ignore const)

$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{d}{dt} m_1 r^2 \dot{\theta} = 0 \Rightarrow l = m_1 r^2 \dot{\theta}$ const.

$L = \frac{1}{2}(m_1+m_2)\dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\theta}^2 - m_2 g r$

$(m_1+m_2)\ddot{r} - m_1 r \dot{\theta}^2 + m_2 g = 0$, $\dot{\theta} = \frac{l}{m_1 r^2}$

$\Rightarrow (m_1+m_2)\ddot{r} - \frac{l^2}{m_1 r^3} + m_2 g = 0$

$l = m_1 r^2 \dot{\theta}$ angular momentum conserved (θ symmetry)

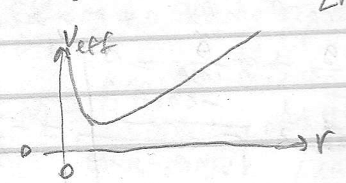
$H = (m_1+m_2)\dot{r}^2 + m_1 r^2 \dot{\theta}^2 - L = \frac{1}{2}(m_1+m_2)\dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\theta}^2 + m_2 g r$

energy conserved (time symmetry)

stationary circular orbit: $\dot{r}=0$, $r=a$

$\frac{l^2}{m_1 a^3} = m_2 g$ i.e. $\sqrt{m_1 m_2 g a^3} = l \ (= m_1 a^2 \dot{\theta})$

$E = \frac{1}{2}(m_1+m_2)\dot{r}^2 + \frac{l^2}{2m_1 r^2} + m_2 g r \Rightarrow V_{eff} = \frac{l^2}{2m_1 r^2} + m_2 g r$



hanging mass pulled: l conserved (L still indep. of θ)

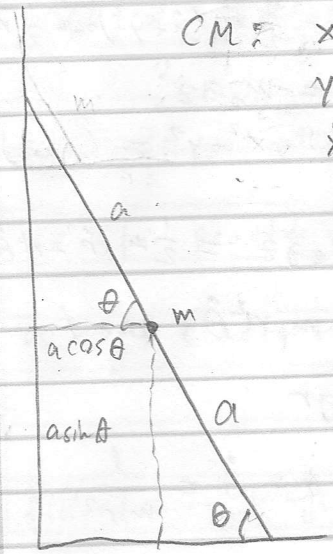
$l = \sqrt{m_1 m_2 g a^3}$, $r = a+q \rightarrow (m_1+m_2)\ddot{q} - \frac{m_2 g a^3}{(a+q)^3} + m_2 g = 0$

$(m_1+m_2)\ddot{q} + m_2 g \left(1 - \left(1 - \frac{3q}{a}\right)\right) = 0$

$\ddot{q} + \frac{3 m_2 g}{(m_1+m_2)a} q = 0 \Rightarrow \omega = \sqrt{\frac{3 m_2 g}{(m_1+m_2)a}}$

circular: $\omega_c = \dot{\theta} = \frac{l}{m_1 a^2} = \sqrt{\frac{m_2 g}{m_1 a}}$, $\frac{\omega}{\omega_c} = \sqrt{\frac{3 m_1}{m_1+m_2}}$

2013 CM 2



CM: $x = a\cos\theta$ $\qquad T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) + \frac{1}{2}I\dot{\theta}^2$

$y = a\sin\theta \qquad I = \frac{1}{3}ma^3$

$\dot{x}^2+\dot{y}^2 = a^2\left(\sin^2\theta\,\dot{\theta}^2 + \cos^2\theta\,\dot{\theta}^2\right) = a^2\dot{\theta}^2$

$\Rightarrow T = \frac{1}{2}ma^2\dot{\theta}^2 + \frac{1}{6}ma^2\dot{\theta}^2 = \frac{2}{3}ma^2\dot{\theta}^2$

$V = mga\sin\theta$

$E = \frac{2}{3}ma^2\dot{\theta}^2 + mga\sin\theta = E_0 = mga\sin\theta_0$

$\Rightarrow \dot{\theta}^2 = \frac{3}{2}\frac{g}{a}(\sin\theta_0 - \sin\theta)$

$\frac{d\theta}{dt} = \sqrt{\frac{3g}{2a}(\sin\theta_0 - \sin\theta)}$

$t = \sqrt{\frac{2a}{3g}}\int_{\theta_0}^{\theta}\frac{d\theta'}{\sqrt{\sin\theta_0 - \sin\theta'}}$

vertical wall constraint is $x = a\cos\theta \Rightarrow dx + a\sin\theta\,d\theta = 0$

$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}ma^2\cos^2\theta\,\dot{\theta}^2 + \frac{1}{6}ma^2\dot{\theta}^2 - mga\sin\theta$

$m\ddot{x} = \lambda, \quad \frac{d}{dt}\left(ma^2\cos^2\theta\,\dot{\theta} + \frac{1}{3}ma^2\dot{\theta}\right)$

$\qquad + ma^2\cos\theta\sin\theta\,\dot{\theta}^2 + mga\cos\theta = a\sin\theta\,\lambda$

θ_1 at $\lambda = 0$,

$ma^2\left(-2\cos\theta\sin\theta\,\dot{\theta}^2 + \cos^2\theta\,\ddot{\theta}\right) + \frac{1}{3}ma^2\ddot{\theta} + ma^2\cos\theta\sin\theta\,\dot{\theta}^2 + mga\cos\theta = 0$

$-2\sin\theta\,\dot{\theta}^2 + \cos\theta\,\ddot{\theta} + \sin\theta\,\dot{\theta}^2 + \frac{g}{a} + \frac{1}{3}\frac{\ddot{\theta}}{\cos\theta} = 0$

but $\dot{\theta} = \sqrt{\frac{3g}{2a}(\sin\theta_0 - \sin\theta)}$, $\quad \ddot{\theta} = \sqrt{\frac{3g}{2a}}\,\frac{1}{2}\,\frac{-\cos\theta\,\dot{\theta}}{\sqrt{\sin\theta_0 - \sin\theta}} = -\frac{3g}{4a}\cos\theta$

$-\frac{3g}{4a}\cos^2\theta - \frac{3g}{2a}\sin\theta(\sin\theta_0 - \sin\theta) + \frac{g}{a} - \frac{g}{4a}$

$\left(\frac{3}{4} + \frac{3}{2}\right)\sin^2\theta + \left(-\frac{3}{2}\sin\theta_0\right)\sin\theta + \cancel{\left(1 - \frac{3}{4} - \frac{1}{4}\right)} = 0$

$\frac{9}{4}\sin^2\theta - \frac{3}{2}\sin\theta_0\,\sin\theta = 0$

$(3\sin\theta - 2\sin\theta_0)\sin\theta \Rightarrow \sin\theta = 0, \ \frac{2}{3}\sin\theta_0$

$\sin\theta_1 = \frac{2}{3}\sin\theta_0$

CM3 $I_i \dot{\omega}_i + \epsilon_{ijk}\omega_j\omega_k I_k = 0$ i.e.

$$I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 = 0 \Rightarrow \ddot{\omega}_1 = \frac{I_2 - I_3}{I_1}\left(\dot{\omega}_2\omega_3 + \omega_2\dot{\omega}_3\right)$$

$$I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 = 0 \qquad \ddot{\omega}_2 = \frac{I_3 - I_1}{I_2}\left(\dot{\omega}_3\omega_1 + \omega_3\dot{\omega}_1\right)$$

$$I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 = 0$$

$$\ddot{\omega}_1 = \frac{I_2 - I_3}{I_1}\left(\frac{I_3 - I_1}{I_2}\omega_3^2\omega_1 + \frac{I_1 - I_2}{I_3}\omega_2^2\omega_1\right)$$

$$= \frac{I_2 - I_3}{I_1}\left(\frac{I_3 - I_1}{I_2}\omega_3^2 + \frac{I_1 - I_2}{I_3}\omega_2^2\right)\omega_1$$

$$\ddot{\omega}_2 = \frac{I_3 - I_1}{I_2}\left(\frac{I_1 - I_2}{I_3}\omega_1^2 + \frac{I_2 - I_3}{I_1}\omega_3^2\right)\omega_2$$

take $\omega_3 \gg \omega_2, \omega_1$

$$\ddot{\omega}_1 \simeq \frac{I_2 - I_3}{I_1}\frac{I_3 - I_1}{I_2}\omega_3^2\omega_1, \quad \ddot{\omega}_2 \simeq \frac{I_3 - I_1}{I_2}\frac{I_2 - I_3}{I_1}\omega_3^2\omega_1$$

$$\Rightarrow \ddot{\omega}_1 - a^2\omega_1 \simeq 0, \quad \ddot{\omega}_2 - a^2\omega_2 \simeq 0, \quad a^2 \equiv \frac{I_2 - I_3}{I_1}\frac{I_3 - I_1}{I_2}$$

I_3 largest $\Rightarrow a^2 < 0 \Rightarrow \omega_{1,2} \sim e^{i|a|t}$, stable

I_3 smallest $\Rightarrow$ same

I_3 middle $\Rightarrow a^2 > 0 \Rightarrow \omega_{1,2} \sim e^{|a|t}$, unstable

CM4 $\frac{\partial Q}{\partial q} = \cos\phi, \quad \frac{\partial Q}{\partial p} = -\sin\phi, \quad \frac{\partial P}{\partial q} = \sin\phi, \quad \frac{\partial P}{\partial p} = \cos\phi$

$$M = \begin{pmatrix}\cos\phi & -\sin\phi \\ \sin\phi & \cos\phi\end{pmatrix}, \quad J = \begin{pmatrix}0 & 1 \\ -1 & 0\end{pmatrix}.$$

$$MJM^T = \begin{pmatrix}\cos\phi & -\sin\phi \\ \sin\phi & \cos\phi\end{pmatrix}\begin{pmatrix}0 & 1 \\ -1 & 0\end{pmatrix}\begin{pmatrix}\cos\phi & \sin\phi \\ -\sin\phi & \cos\phi\end{pmatrix} = \begin{pmatrix}\cos\phi & -\sin\phi \\ \sin\phi & \cos\phi\end{pmatrix}\begin{pmatrix}-\sin\phi & \cos\phi \\ -\cos\phi & -\sin\phi\end{pmatrix}$$

$$= \begin{pmatrix}0 & 1 \\ -1 & 0\end{pmatrix} \Rightarrow \text{canonical for all } \phi.$$

2013

$$p = \frac{\partial F_1(q,Q)}{\partial q} = q\frac{\cos\phi}{\sin\phi} - Q\frac{1}{\sin\phi} \Rightarrow F_1 = \frac{q^2}{2}\frac{\cos\phi}{\sin\phi} - \frac{qQ}{\sin\phi} + f(Q)$$

$$-P = \frac{\partial F_1}{\partial Q} = -q\sin\phi - q\frac{\cos^2\phi}{\sin\phi} + Q\frac{\cos\phi}{\sin\phi} = -\frac{q}{\sin\phi} + Q\frac{\cos\phi}{\sin\phi}$$

$$= -\frac{q}{\sin\phi} + \frac{\partial f}{\partial Q} \Rightarrow f(Q) = \frac{Q^2}{2}\frac{\cos\phi}{\sin\phi} + \cancel{c} \text{ ignore}$$

$$F_1(q,Q) = \frac{q^2}{2}\frac{\cos\phi}{\sin\phi} - \frac{qQ}{\sin\phi} + \frac{Q^2}{2}\frac{\cos\phi}{\sin\phi}$$

$$= \frac{(q^2+Q^2)}{2}\cot\phi - qQ\csc\phi$$

$$H = P^2 + Q^2 = q^2 + p^2 + 2pq\sin\cancel{\phi}\cos\phi - 2pq\cos\cancel{\phi\sin\phi}$$

$$= p^2 + q^2$$

H is cyclic in ϕ. Period of ϕ is 2π so take $\frac{\phi}{2\pi} = w$, angle variable.

$$J = \oint p\,dq = \oint\sqrt{H-q^2}\,dq = \sqrt{H}\oint\sqrt{1-\left(\tfrac{q}{\sqrt{H}}\right)^2}\,\sqrt{H}\,d\left(\tfrac{q}{\sqrt{H}}\right)$$

$$= H\oint dx\sqrt{1-x^2}, \quad x = \sin\theta \quad dx = \cos\theta\,d\theta$$

$$= H\oint\cos^2\theta\,d\theta = \pi H$$

CM 5 $\dot{x}^2 = l^2(2\dot{\phi} + 2\cos(2\phi)\dot{\phi})^2 \qquad \dot{z}^2 = l^2(2\sin(2\phi)\dot{\phi})^2$

$$\dot{x}^2 + \dot{z}^2 = 4l^2\left[\dot{\phi}^2 + 2\dot{\phi}^2\cos(2\phi) + \dot{\phi}^2\right) = 8l^2\dot{\phi}^2\underbrace{(1+\cos(2\phi))}_{2\cos^2\phi}$$

$$T = 4ml^2\dot{\phi}^2(1+\cos(2\phi)), \quad V = mgz = mgl\underbrace{(1-\cos(2\phi))}_{2\sin^2\phi}$$

$$L = 4ml^2\dot{\phi}^2(1+\cos(2\phi)) - mgl(1-\cos 2\phi)$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = 8ml^2\dot{\phi}(1+\cos 2\phi) \Rightarrow H = 4ml^2\dot{\phi}^2(1+\cos(2\phi)) + mgl(1-\cos 2\phi)$$

$$H = \frac{p_\phi^2}{16ml^2(1+\cos 2\phi)} + mgl(1-\cos 2\phi)$$

$$p_\phi = \frac{\partial S}{\partial \phi} \rightarrow \frac{1}{16ml^2(1+\cos 2\phi)}\left(\frac{\partial S}{\partial \phi}\right)^2 + mgl(1-\cos 2\phi) + \underbrace{\frac{\partial S}{\partial t}}_{-E} = 0$$

$$\left(\frac{\partial S}{\partial \phi}\right)^2 = 16ml^2(1+\cos 2\phi)\left[-mgl(1-\cos 2\phi) + E\right] = 0$$

$$S = 4l\sqrt{m}\int d\phi \sqrt{(1+\cos 2\phi)(E - mgl(1-\cos 2\phi))} - Et \qquad \left(\frac{\partial S}{\partial E} = \beta\right)$$

$$4l\sqrt{m}\int d\phi \sqrt{(1+\cos 2\phi)}\,\frac{1}{2}\,\frac{1}{\sqrt{E - mgl(1-\cos 2\phi)}} = \beta + t$$

$$\Rightarrow \int d\phi \sqrt{\frac{1+\cos 2\phi}{E - mgl(1-\cos 2\phi)}} = \frac{t+\beta}{2l\sqrt{m}} = \sqrt{2}\int \frac{\cos\phi\, d\phi}{\sqrt{E - 2mgl\sin^2\phi}}$$

$$= \sqrt{\frac{2}{E}}\int \frac{du}{\sqrt{1-au^2}} = \sqrt{\frac{2}{Ea}}\int \frac{dv}{\sqrt{1-v^2}} = \sqrt{\frac{1}{mgl}}\sin^{-1}\left(\sqrt{\frac{2mgl}{E}}\sin\phi\right)$$

$u = \sin\phi, \; a = \frac{2mgl}{E}, \; v = \sqrt{a}\,u \Rightarrow \sqrt{\frac{2mgl}{E}}\sin\phi = \sin\left(\frac{1}{2}\sqrt{\frac{g}{l}}(t+\beta)\right)$

→ oscillates for $E < 2mgl$, bifurcates for $E = 2mgl$, invalid $E > 2mgl$

$$J = \oint \sqrt{16ml^2(1+\cos 2\phi)}\sqrt{E - mgl(1-\cos 2\phi)}\, d\phi \qquad \sin\theta = \sqrt{\frac{2mgl}{E}}\sin\phi$$

$$\cos\theta\, d\theta = \sqrt{\frac{2mgl}{E}}\cos\phi\, d\phi$$

$$= 4l\sqrt{2mE}\oint \cos\phi\, d\phi \sqrt{1 - \frac{2mgl}{E}\sin^2\phi} = \frac{4lE}{\sqrt{gl}}\oint \cos^2\theta\, d\theta = 4\pi E\sqrt{\frac{l}{g}}$$

adiabatic: $J = 4\pi E\sqrt{\frac{l}{g}}$ constant $\Rightarrow E l^{1/2} = \text{const} \Rightarrow E \propto l^{-1/2}$

2013 SM1 $a(T) = \sum_{\text{states } s} e^{-\beta E_s}$, $Q(N,T) = \frac{1}{?} \sum_{\text{config } r} e^{-\beta E_r} \sim a(T)^{N_{ad}}$ (non-interacting)

$\uparrow$ all single particle states, N_{ad} particles

note: $\frac{N_0!}{N_{ad}!(N_0-N_{ad})!}$ ways to fill sites, so weight config with N_{ad} particles by this.

$$Q(N,T) = \frac{N_0!}{N_{ad}!(N_0-N_{ad})!} a(T)^{N_{ad}}$$

$$A = -kT\log Q = -kT\Big(N_0\log N_0 - \cancel{N_0} - N_{ad}\log N_{ad} + \cancel{N_{ad}} - (N_0-N_{ad})\log(N_0-N_{ad}) + (\cancel{N_0 - N_{ad}}) + N_{ad}\log a(T)\Big)$$

$$= kT\Big[N_0\big(\log(N_0-N_{ad}) - \log N_0\big) + N_{ad}\big(\log N_{ad} - \log(N_0-N_{ad}) - \log a\big)\Big]$$

$$\mu_{ad} = \frac{\partial A}{\partial N_{ad}} = kT\left[\frac{\cancel{-N_0}}{N_0-N_{ad}} + \log N_{ad} - \log(N_0-N_{ad}) - \log a + \cancel{1} + \cancel{\frac{N_{ad}}{N_0-N_{ad}}}\right]$$

$$= kT\log\left[\frac{N_{ad}}{(N_0-N_{ad})a}\right] = \mu_{gas} = kT\log\left[\frac{N-N_{ad}}{V}\left(\frac{h^2}{2\pi mkT}\right)^{3/2}\right]$$

$$\Rightarrow \frac{N-N_{ad}}{V}\left(\frac{h^2}{2\pi mkT}\right)^{3/2} = \frac{N_{ad}}{(N_0-N_{ad})a(T)}$$

SM2 $\Phi = -PV = kT\sum_{\varepsilon}\log(1-e^{-\varepsilon/kT}) = kT\int d\varepsilon\, a(\varepsilon)\log(1-e^{-\varepsilon/kT})$

$$a(\varepsilon) = \frac{V}{h^3}\frac{8\pi}{3}\frac{1}{c^3}3\varepsilon^2 = 8\pi\frac{V}{c^3h^3}\varepsilon^2$$

$$\Phi = \frac{8\pi V}{h^3c^3}kT\int d\varepsilon\,\varepsilon^2\log(1-e^{-\varepsilon/kT})$$

$$S = -\frac{\partial\Phi}{\partial T}\bigg|_{\mu,V} = \frac{-8\pi V}{h^3c^3}k\int\varepsilon^2 d\varepsilon\left[\log(1-e^{-\varepsilon/kT}) - T\frac{\frac{\varepsilon}{kT^2}e^{-\varepsilon/kT}}{1-e^{-\varepsilon/kT}}\right]$$

$$= \frac{-8\pi V}{(hc)^3}k\left[\underbrace{\int_0^\infty\varepsilon^2 d\varepsilon\log(1-e^{-\varepsilon/kT})}_{I_1} - \frac{1}{kT}\underbrace{\int_0^\infty\frac{\varepsilon^3 d\varepsilon}{e^{\varepsilon/kT}-1}}_{I_2}\right]$$

$$I_1 = -\int_0^\infty \frac{1}{3}\varepsilon^3 d\varepsilon \frac{\frac{1}{kT}e^{-\varepsilon/kT}}{1-e^{-\varepsilon/kT}} = \frac{-1}{3kT}\int_0^\infty \frac{\varepsilon^3 d\varepsilon}{e^{\varepsilon/kT}-1} = \frac{-I_2}{3kT}$$

$x = \varepsilon/kT \Rightarrow \varepsilon = kTx$

$$S = \frac{32\pi V}{3(hc)^3}\frac{I_2}{T}, \quad I_2 = (kT)^4\frac{\pi^4}{15}$$

$$S = \frac{32\pi^5 k^4}{45(hc)^3} V T^3$$

SM3 $Q_1 = 1 + e^{-\beta\varepsilon} + e^{-2\beta\varepsilon}$, noninteracting + distinguishable $\Rightarrow$

$$Q_N = (1+e^{-\beta\varepsilon}+e^{-2\beta\varepsilon})^N$$

$$A = -kT\log Q_N = -NkT\log(1+e^{-\beta\varepsilon}+e^{-2\beta\varepsilon}), \quad \beta = \frac{1}{kT}$$

$$U = -\frac{\partial}{\partial\beta}\log Q_N = -N\frac{\partial}{\partial\beta}\log(1+e^{-\beta\varepsilon}+e^{-2\beta\varepsilon}) = -N\frac{-\varepsilon e^{-\beta\varepsilon}-2\varepsilon e^{-2\beta\varepsilon}}{1+e^{-\beta\varepsilon}+e^{-2\beta\varepsilon}}$$

$$= N\varepsilon\frac{e^{-\beta\varepsilon}+2e^{-2\beta\varepsilon}}{1+e^{-\beta\varepsilon}+e^{-2\beta\varepsilon}}, \quad \beta = \frac{1}{kT}$$

$$A = U - TS \Rightarrow S = \frac{1}{T}(U-A) = N\left[\frac{\varepsilon}{T}\frac{e^{-\beta\varepsilon}+2e^{-2\beta\varepsilon}}{1+e^{-\beta\varepsilon}+e^{-2\beta\varepsilon}} + k\log(1+e^{-\beta\varepsilon}+e^{-2\beta\varepsilon})\right]$$

$$C_V = \frac{\partial U}{\partial T} = \frac{d\beta}{dT}\frac{\partial U}{\partial\beta} = \frac{-1}{kT^2}\frac{\partial U}{\partial\beta}$$

$$= -\frac{1}{kT^2}\frac{(-\varepsilon e^{-\beta\varepsilon}-4\varepsilon e^{-2\beta\varepsilon})(1+e^{-\beta\varepsilon}+e^{-2\beta\varepsilon}) - (e^{-\beta\varepsilon}+2e^{-2\beta\varepsilon})(-\varepsilon e^{-\beta\varepsilon}-2\varepsilon e^{-2\beta\varepsilon})}{(1+e^{-\beta\varepsilon}+e^{-2\beta\varepsilon})^2}$$

$$= \frac{\varepsilon}{kT^2}\frac{e^{-\beta\varepsilon}+\cancel{e^{-2\beta\varepsilon}}+e^{-3\beta\varepsilon}+4e^{-2\beta\varepsilon}+4e^{-3\beta\varepsilon}+\cancel{4e^{-4\beta\varepsilon}}-\cancel{e^{-2\beta\varepsilon}}-2\cancel{e^{-3\beta\varepsilon}}-2\cancel{e^{-3\beta\varepsilon}}-\cancel{4e^{-4\beta\varepsilon}}}{(\ldots)}$$

$$= \frac{\varepsilon}{kT^2}\frac{e^{-\beta\varepsilon}(1+4e^{-\beta\varepsilon}+e^{-2\beta\varepsilon})}{(1+e^{-\beta\varepsilon}+e^{-2\beta\varepsilon})^2}, \quad \beta = \frac{1}{kT}$$

2013 SM4 $NL = \sum_{i=1}^{N} \sigma_i$. $\sum_{n.n.} \sigma_i \sigma_j = \frac{1}{q} \sum_{i=1}^{N} \sigma_i \sum_{j=1, n.n.}^{q} \sigma_j = N_{++} + N_{--} - N_{+-}$

note $N_{++} + N_{--} + N_{+-} = \frac{q}{2} N$, $\left. \begin{array}{l} 2N_{--} + N_{+-} = qN_- \\ 2N_{++} + N_{+-} = qN_+ \end{array} \right\}$

$\Rightarrow N_{+-} = qN_+ - 2N_{++}$

$N_{--} = \frac{1}{2}(qN_- - N_{+-}) = \frac{q}{2} N_- - \frac{q}{2} N_+ + N_{++}$

$\Rightarrow \sum_{nn} \sigma_i \sigma_j = 4N_{++} + \frac{q}{2}(N_- - N_+) - qN_+$

$= 4N_{++} - \frac{q}{2} N + q(N_- - N_+) = 4N_{++} - qNL - \frac{1}{2} qN$

$\Rightarrow H = -J\left(4N_{++} - qNL - \frac{1}{2} qN\right) - B\mu NL$

takes ratio of ++ pairs to total pairs as the same as probability that any two selected sites are +, i.e. the + sites are randomly distributed.

$u = \overline{H} = -J\left(2qN\overline{\left(\frac{N_+}{N}\right)^2} - qN\overline{L} - \frac{1}{2} qN\right) - B\mu N\overline{L}$

$NL = N_+ - N_- = 2N_+ - N \Rightarrow (NL)^2 = 4N_+^2 - 4N_+ N + N^2$
$= 4N_+^2 - N(2NL + N)$

$N_+^2 = \frac{1}{4} N^2 L^2 + \frac{1}{2} N^2 L + \frac{1}{4} N^2$

$u = -qJN\left(\frac{1}{2}\overline{L^2} + \overline{L} + \frac{1}{2} - \overline{L} - \frac{1}{2}\right) - B\mu N\overline{L}$

$= -\frac{1}{2} qJN\overline{L^2} - \mu BN\overline{L}$

SM5 $Q_N = \sum_r e^{-\beta E_r} = \sum_{\{n_\varepsilon\}} e^{-\beta \sum_\varepsilon n_\varepsilon \varepsilon}$

all $n_\varepsilon = 0$ or 1, $\sum n_\varepsilon = N$

$\mathcal{Q} = \sum_{N=0}^{\infty} \sum_{\{n_\varepsilon\}} z^N e^{-\beta \sum_\varepsilon n_\varepsilon \varepsilon} = \sum_{\{n_\varepsilon\}} z^{\sum_\varepsilon n_\varepsilon} e^{-\beta \sum_\varepsilon n_\varepsilon \varepsilon}$

all $n_\varepsilon = 0$ or 1, $\sum n_\varepsilon = N$; 0 or 1

$= \sum_{\{n_\varepsilon\}} \prod_\varepsilon \left(z e^{-\beta \varepsilon}\right)^{n_\varepsilon} = \prod_\varepsilon \left(1 + z e^{-\beta \varepsilon}\right)$, $z = e^{\beta\mu}$

$\mathcal{Q} = \prod_p \left(1 + e^{-\beta(\varepsilon_p - \mu)}\right)$

$$\mathcal{Q} = \sum_N \sum_{\substack{r \\ \text{config}}} e^{\beta\mu N - \beta E_r}, \quad N = \frac{\frac{1}{\beta}\partial \mathcal{Q}/\partial\mu}{\mathcal{Q}} = \frac{1}{\beta}\frac{d}{d\mu}\log \mathcal{Q}$$

$$N = \frac{1}{\beta}\sum_p \frac{\beta e^{-\beta(\varepsilon_p-\mu)}}{1+e^{-\beta(\varepsilon_p-\mu)}} = \sum_p \frac{1}{e^{\beta(\varepsilon_p-\mu)}+1}$$

$$\langle n_p \rangle = \frac{1}{e^{\beta(\varepsilon_p-\mu)}+1} \longrightarrow \frac{1}{2}, \quad T\to\infty$$

$$\longrightarrow \begin{cases} 0, & \varepsilon_p > \mu \\ 1, & \varepsilon_p < \mu \end{cases} \quad \text{for } T\to 0$$

$$E - \mu N = -\frac{\partial}{\partial\beta}\log \mathcal{Q} = \sum_p \frac{(\varepsilon_p-\mu)e^{-\beta(\varepsilon_p-\mu)}}{1+e^{-\beta(\varepsilon_p-\mu)}} = \sum_p \frac{\varepsilon_p}{e^{\beta(\varepsilon_p-\mu)}+1} - \mu N$$

$$PV = kT\sum_p \log\left(1+e^{-\beta(\varepsilon_p-\mu)}\right), \quad \sum_p \to \frac{V}{h^3}\int 4\pi p^2 dp$$

$$= kT\frac{V}{h^3}4\pi\int_0^\infty p^2 dp \log\left(1+e^{-\beta\left(\frac{p^2}{2m}-\mu\right)}\right)$$

$$= -kT\frac{V}{h^3}4\pi\int_0^\infty \frac{1}{3}p^3 dp \frac{-\beta/m\, p\, e^{-\beta\left(\frac{p^2}{2m}-\mu\right)}}{1+e^{-\beta\left(\frac{p^2}{2m}-\mu\right)}}$$

$$= \frac{2}{3}\frac{V}{h^3}\int_0^\infty 4\pi p^2 dp \frac{\frac{p^2}{2m}}{e^{\beta\left(\frac{p^2}{2m}-\mu\right)}+1} = \frac{2}{3}\sum_p \frac{\varepsilon_p}{e^{\beta(\varepsilon_p-\mu)}+1} = \frac{2}{3}E$$

$$PV = kT\frac{V}{h^3}4\pi\int_0^\infty p^2 dp \underbrace{\log\left(1+ze^{-p^2/2m}\right)}_{ze^{-p^2/2m} - \frac{1}{2}z^2 e^{-p^2/m} + \mathcal{O}(z^3)}$$

$$= c_0 + c_1 z + c_2 z^2 + \mathcal{O}(z^3)$$

$$c_0 = 0, \quad c_1 = kT\frac{V}{h^3}4\pi\int_0^\infty p^2 e^{-p^2/2m}dp = \frac{4\pi kTV}{h^3}(2m)^{3/2}\int_0^\infty x^2 e^{-x^2}dx$$

$$c_2 = -\frac{2\pi kTV}{h^3}\int_0^\infty p^2 e^{-p^2/m}dp = -\frac{2\pi kTV}{h^3}m^{3/2}\int_0^\infty x^2 e^{-x^2}dx$$

where $\int_0^\infty x^2 e^{-x^2}dx = -\frac{1}{2}\int_0^\infty x\, d(e^{-x^2}) = \frac{1}{2}\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{4}$

2013 EM1-1 $\nabla^2 \Phi = -\frac{\rho}{\epsilon_0}, \quad \nabla^2 G(\vec{x},\vec{x}') = -4\pi\delta^3(\vec{x}-\vec{x}')$

$$\int_V \Phi \nabla^2 G \, dV' = -4\pi \Phi(\vec{x}) = \oint_{\partial V} \Phi \nabla G \cdot d\vec{s}' - \int_V \nabla\Phi \cdot \nabla G \, dV'$$

$$= \oint_{\partial V} \Phi \nabla G \cdot d\vec{s} - \oint_{\partial V} G \nabla\Phi \cdot d\vec{s} + \underbrace{\int_V \nabla^2 \Phi \, G \, dV'}_{\int_V -\rho/\epsilon_0 \, G \, dV'}$$

$$\Rightarrow \Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{r}') \, G(\vec{r},\vec{r}') \, dV'$$

$$+ \frac{1}{4\pi}\left[\oint_{\partial V} G(\vec{x},\vec{x}') \nabla\Phi(\vec{x}') \cdot d\vec{s}' - \oint_{\partial V} \Phi(\vec{x}') \nabla G(\vec{x},\vec{x}') \cdot d\vec{s}'\right]$$

V_0, V_1:

$G(\vec{r},\vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|}$, take plane as xy-plane

$$\Phi(\vec{r}) = \frac{-1}{4\pi}\int_S \Phi(\vec{r}') \nabla' G(\vec{r},\vec{r}') \cdot d\vec{s}$$

$$\nabla' G = \sum_i \hat{i} \, \partial_{i'} \frac{1}{\left(\sum_j (x_j - x_j')^2\right)^{1/2}} = -\frac{1}{2}\frac{-2(x_i - x_i')}{|\vec{x}-\vec{x}'|^3} = \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}$$

$$\Phi(\vec{r}) = \frac{-V_0 z}{4\pi}\int_{S_0}\frac{ds'}{|\vec{x}-\vec{x}'|^3} - \frac{V_1 z}{4\pi}\int_{S_1}\frac{ds'}{|\vec{x}-\vec{x}'|^3}$$

S_0 = plane
S_1 = hole

W_0, W_1:

$$\Phi(\vec{r}) = \frac{1}{4\pi}\int_S \frac{\nabla\Phi \cdot d\vec{s}'}{|\vec{x}-\vec{x}'|} = \frac{W_0}{4\pi}\int_{S_0}\frac{ds'}{|\vec{x}-\vec{x}'|} + \frac{W_1}{4\pi}\int_{S_1}\frac{ds'}{|\vec{x}-\vec{x}'|}$$

EM1-2 $\nabla\times\vec{H} = \frac{\partial \vec{D}}{\partial t} = 0 \Rightarrow H = -\nabla\Phi_M$.

$$\nabla\cdot\vec{B} = \nabla\cdot(\mu\vec{H}) = \nabla\mu\cdot\vec{H} + \mu\nabla\cdot\vec{H} = -\nabla\mu\cdot\nabla\Phi_M - \mu\nabla^2\Phi_M = 0$$

$$\nabla\mu\cdot\nabla\Phi_M + \mu\nabla^2\Phi_M = 0$$

$$\nabla\cdot\vec{B} = \mu_0\nabla\cdot(\vec{H}+\vec{M}) = \mu_0\nabla\cdot\vec{H} + \mu_0\nabla\cdot\vec{M} = 0$$

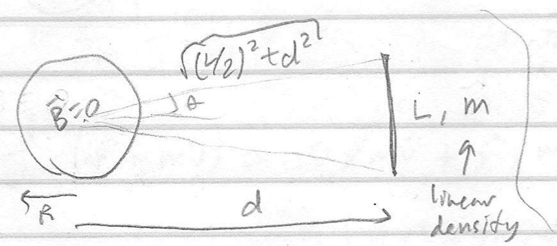
$$\Rightarrow \nabla^2\Phi_M = \nabla\cdot\vec{M}$$

$\vec{J} = -\frac{1}{\mu_0 \lambda_L^2}\vec{A}$, $\nabla \times \vec{H} = \vec{J}$, $\nabla \cdot \vec{B} = 0 \Rightarrow \vec{B} = \nabla \times \vec{A}$

$$\vec{B} = -\mu_0 \lambda_L^2 \nabla \times \vec{J} = -\mu_0 \lambda_L^2 \underbrace{\nabla \times (\nabla \times \vec{H})}_{\nabla(\nabla\cdot\vec{H}) - (\nabla\cdot\nabla)\vec{H}} = -\mu_0 \lambda_L^2 \left(\nabla(\nabla\cdot\vec{H}) - \nabla^2 \vec{H}\right)$$

μ constant (homogeneous) $\Rightarrow \nabla \cdot \vec{H} = \frac{1}{\mu}\nabla\cdot\vec{B} = 0 \Rightarrow$

$\vec{B} = +\mu_0 \lambda_L^2 \frac{1}{\mu}\nabla^2 \vec{B} = \frac{\mu_0}{\mu}\lambda_L^2 \nabla^2 \vec{B} \Rightarrow \lambda_L \sqrt{\frac{\mu_0}{\mu}}$ is penetration length



$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M} \Rightarrow \nabla\cdot\vec{H} = -\nabla\cdot\vec{M}$

$\nabla^2 \vec{H} = \frac{1}{\mu_0}\nabla^2\vec{B} - \nabla^2 \vec{M}$

$\Rightarrow \vec{B} = \lambda_L^2\left(\nabla^2\vec{B} - \mu_0 \nabla^2 \vec{M} + \mu_0 \nabla(\nabla\cdot\vec{M})\right)$

pick origin at center of sphere.

$$\vec{M}(\vec{x}) = \begin{cases} m\,\delta(x-d)\,\delta(y)\,\hat{z}, & z \in [-R,R] \\ 0 & \text{otherwise} \end{cases}$$

$$= m\,\delta(x-d)\,\delta(y)\left[\theta(z+R) - \theta(z-R)\right]\hat{z}$$

equivalent to two monopoles of charge $q_m = \pm\frac{m}{L}$ at $x=d$, $z = \pm\frac{L}{2}$

$\Rightarrow$ image charges at $r = \frac{R^2}{\sqrt{(\frac{L}{2})^2 + d^2}}$, $\theta = \pm\tan^{-1}\left(\frac{L}{2d}\right)$

$\cos\theta = \frac{d}{\sqrt{d^2+(L/2)^2}}$ $\quad \sin\theta = \pm\frac{L/2}{\sqrt{d^2+(L/2)^2}}$

$$\Rightarrow x = r\cos\theta = \frac{R^2 d}{d^2 + (\frac{L}{2})^2}, \quad z = \pm \frac{R^2 L/2}{d^2+(\frac{L}{2})^2}$$

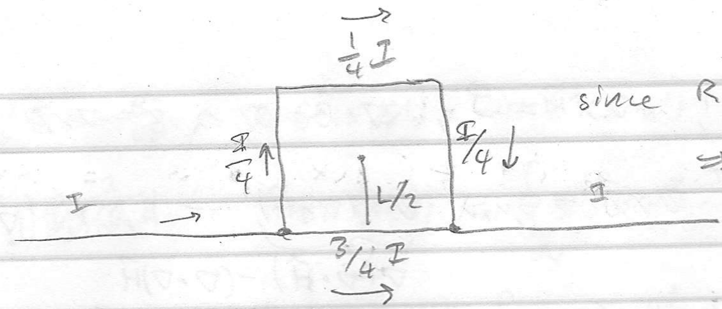
$$q_m' = \mp \frac{R}{r} q_m = \mp \frac{\sqrt{d^2+(L/2)^2}}{R}\frac{m}{L}$$

$$\Phi_M(\vec{x}) = \sum \frac{q_m}{4\pi|x-x'|} = \frac{1}{4\pi}\frac{m}{L}\left[\frac{1}{\sqrt{(x-d)^2+(z-\frac{L}{2})^2+y^2}} - \frac{1}{\sqrt{(x-d)^2+(z-L/2)^2+y^2}}\right.$$

$$\left. + \frac{\sqrt{d^2+(L/2)^2}}{R}\left(\frac{-1}{\sqrt{(x-x')^2+(z-z')^2+y^2}} + \frac{1}{\sqrt{(x-x')^2+(z+z')^2+y^2}}\right)\right]$$

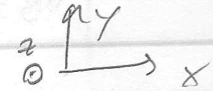
$$x' = \frac{R^2 d}{d^2+(\frac{L}{2})^2} \quad z' = \frac{R^2 L/2}{d^2+(\frac{L}{2})^2}$$

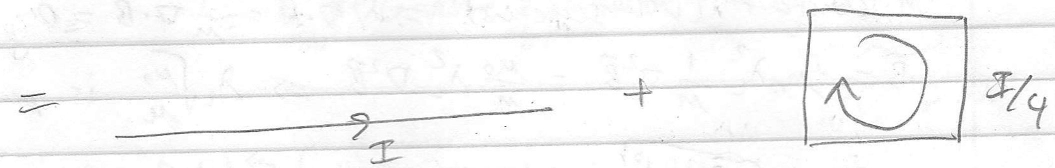
2013 EM1-3



since $R_{upper} = 3R_{lower}$

$\Rightarrow I_{lower} = 3 I_{upper}$





$$\vec{B}_{line} = \frac{\mu_0 I}{2\pi (L/2)}\,\hat{z}$$

$$\vec{B}_{loop} = 4\,\vec{B}_{segment}$$

$$\vec{B}_{segment} = \frac{\mu_0}{4\pi}\int_{-L/2}^{L/2} -\frac{I}{4}\,\frac{d\vec{x}\times\left(-x\hat{x}+\frac{L}{2}\hat{y}\right)}{\left(x^2+\left(\frac{L}{2}\right)^2\right)^{3/2}} = -\frac{I}{4}\frac{L}{2}\,\hat{z}\,\frac{\mu_0}{4\pi}\int_{-L/2}^{L/2}\frac{dx}{\left(x^2+\left(\frac{L}{2}\right)^2\right)^{3/2}}$$

$$\tan\theta = \frac{2}{L}x \Rightarrow dx = \frac{L}{2}\sec^2\theta\, d\theta$$

$$\vec{B}_{loop} = -\frac{\mu_0}{4\pi}\frac{IL}{2}\hat{z}\left(\frac{2}{L}\right)^2 \underbrace{\int_{-\pi/4}^{\pi/4}\frac{\sec^2\theta\, d\theta}{\sec^3\theta}}$$

$$\int\cos\theta\, d\theta = \sin\theta\Big|_{-\pi/4}^{\pi/4} = \sqrt{2}$$

$$= -\frac{\mu_0}{4\pi}\sqrt{2}\,\frac{I\hat{z}}{(L/2)}$$

$$\Rightarrow \vec{B} = \frac{\mu_0 I\hat{z}}{\pi L}\left(1-\frac{\sqrt{2}}{2}\right)$$

EM1-5 $\int_V \nabla\cdot\vec{B}\,dV = \int_V 4\pi\delta(\vec{r}) = 4\pi$

but $\nabla\cdot\vec{B} = \nabla\cdot(\nabla\times\vec{A}) = 0 \Rightarrow \int_V \nabla\cdot\vec{B} = 0$

$\Rightarrow$ cannot have $\nabla\times\vec{A} = \vec{B}$ everywhere

$$\vec{B} = \nabla \times \vec{A} + C\,\delta(x)\delta(y)\theta(z)\,\hat{z}$$

$$\nabla\cdot\vec{B} = C\,\delta(x)\delta(y)\,\partial_z\theta(z) = C\,\delta(x)\delta(y)\delta(z) = 4\pi\,\delta(\vec{r})$$

$$\Rightarrow C = 4\pi$$

$$\bullet g \qquad \overset{m,e}{\circ}\!\longrightarrow \vec{v} \qquad m\frac{d\vec{v}}{dt} = \frac{e}{c}\vec{v}\times\vec{B}, \quad \vec{B} = \nabla\times\vec{A} + 4\pi\,\delta(x)\delta(y)\theta(z)\,\hat{z} = \frac{g}{r^2}\hat{r}$$

$$m\frac{d\vec{v}}{dt} = \frac{eg}{cr^2}\vec{v}\times\hat{r}$$, (r,θ,φ) spherical coords

$$\frac{d}{dt}(\vec{r}\times m\vec{v}) = \overbrace{\cancel{\vec{v}\times m\vec{v}}}^{0} + \vec{r}\times m\frac{d\vec{v}}{dt} = \frac{eg}{cr^3}\vec{r}\times(\vec{v}\times\vec{r})$$

$$= \frac{eg}{cr^3}\left(\vec{v}r^2 - \vec{r}(\vec{v}\cdot\vec{r})\right) = \frac{eg}{c}\left(\frac{\vec{v}}{r} - \frac{\vec{r}\,\overbrace{(\vec{v}\cdot\vec{r})}^{rv_r}}{r^3}\right)$$

but $\frac{d}{dt}\left(\frac{\vec{r}}{r}\right) = \frac{\vec{v}}{r} - \frac{\vec{r}\,v_r}{r^2} \Rightarrow \frac{d}{dt}(\vec{r}\times m\vec{v}) = \frac{eg}{c}\frac{d}{dt}\left(\frac{\vec{r}}{r}\right)$

$$\Rightarrow \frac{d\vec{J}}{dt} = 0, \quad \vec{J} = \vec{r}\times m\vec{v} - \frac{eg}{c}\frac{\vec{r}}{r}$$

$\vec{J}$ quantized in $\hbar$, take straight line motion $\vec{v}\times\hat{r} = 0$

$$\Rightarrow m\frac{d\vec{v}}{dt} = 0.$$

$$\vec{J} = -\frac{eg}{c}\frac{\vec{r}}{r} \Rightarrow J_r = -\frac{eg}{c} = n\hbar, \quad n\in\mathbb{Z}$$

$$\Rightarrow \frac{eg}{c} = n\hbar, \quad n\in\mathbb{Z}$$

EMI-4 use image charges

2013 QM1-1 $[J_i, A_j] = i\hbar \epsilon_{ijk} A_k$

$$[J_i, A_j B_j] = J_i A_j B_j - A_j B_j J_i$$
$$= [J_i, A_j] B_j + A_j [J_i, B_j]$$
$$= i\hbar \epsilon_{ijk} A_k B_j + i\hbar \epsilon_{ijk} A_j B_k = -i\hbar \underbrace{\epsilon_{ijk}}_{\text{anti sym}} \underbrace{(A_k B_j + A_j B_k)}_{\text{sym}}$$
$$= 0$$

$\Rightarrow A_j B_j = \vec{A} \cdot \vec{B}$ scalar

$$[J_i, \epsilon_{jkl} A_k B_l] = \epsilon_{jkl} (J_i A_k B_l - A_k B_l J_i)$$
$$= \epsilon_{jkl} ([J_i, A_k] B_l + A_k [J_i, B_l])$$

$\left(\overset{?}{=} i\hbar \epsilon_{ijm} \epsilon_{mkl} A_k B_l\right)$

$$= \epsilon_{jkl} (-i\hbar \epsilon_{ikm} A_m B_l - i\hbar \epsilon_{ilm} A_k B_m)$$
$$= i\hbar \epsilon_{jkl} (\epsilon_{ikm} A_m B_l + \epsilon_{ilm} A_k B_m)$$
$$= i\hbar \left(\underbrace{\epsilon_{klj} \epsilon_{kmi}}_{\delta_{lm}\delta_{ji} - \delta_{li}\delta_{jm}} A_m B_l + \underbrace{\epsilon_{ljk} \epsilon_{lmi}}_{\delta_{jm}\delta_{ki} - \delta_{ji}\delta_{km}} A_k B_m \right)$$
$$= i\hbar \left(\cancel{A_l B_l \delta_{ij}} - A_j B_i + A_i B_j - \cancel{A_k B_k \delta_{ij}} \right)$$
$$= i\hbar \underbrace{(\delta_{ik}\delta_{lj} - \delta_{il}\delta_{kj})}_{\epsilon_{ijm}\epsilon_{mkl}} A_k B_l = i\hbar \epsilon_{ijk} (\vec{A} \times \vec{B})_k$$

QM1-2 $\oint p\,dx = nh$, $p = \sqrt{2m(E-V)}$

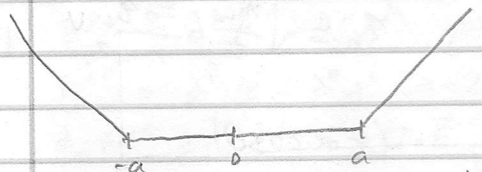
$$\int_{x_1}^{x_2} \sqrt{2m[E - V(x)]}\, dx = (n + \tfrac{1}{2})\pi\hbar$$

$$\int_{x_1}^{x_2} \sqrt{2m(E - \alpha x^4)}\, dx = \sqrt{2mE} \int_{x_1}^{x_2} \sqrt{1 - \frac{\alpha}{E} x^4}\, dx, \quad y = \left(\frac{\alpha}{E}\right)^{1/4} x$$

$$(n + \tfrac{1}{2})\pi\hbar = \sqrt{2mE} \left(\frac{E}{\alpha}\right)^{1/4} 2 \int_0^1 \sqrt{1 - y^4}\, dy, \quad \int_0^1 \sqrt{1-y^4}\, dy \equiv I$$

$$\left(\frac{4m^2}{\alpha}\right)^{1/4} E^{3/4}\, 2I = (n + \tfrac{1}{2})\pi\hbar$$

$$E = \left(\frac{\alpha}{4m^2}\right)^{1/3} \left[\frac{(n+\frac{1}{2})\pi\hbar}{2I}\right]^{4/3} \quad \text{where } I \equiv \frac{\Gamma(1/4)\Gamma(3/2)}{4\Gamma(7/4)}$$



$$(n+\tfrac{1}{2})\pi\hbar = 2\left[\int_0^a + \int_a^{x_2}\right] dx \sqrt{2m(E-V(x))}$$

$$= 2a\sqrt{2mE} + 2\int_a^{x_2} \sqrt{2m(E-k(x-a))}\, dx$$

$y_2 = x_2 - a$

$$= 2\sqrt{2mE}\left[a + \int_0^{y_2} \sqrt{1-\tfrac{k}{E}y}\, dy\right] \longrightarrow \tfrac{2}{3}\left(1-\tfrac{k}{E}y\right)^{3/2}\left(-\tfrac{E}{k}\right)$$

$$= 2\sqrt{2mE}\left[a - \frac{2E}{3k}\left(1-\frac{k}{E}y\right)^{3/2}\Big|_0^{y_2}\right], \qquad \frac{k}{E}y_2 = 1$$

$$= 2\sqrt{2mE}\left[a + \frac{2E}{3k}\right] = (n+\tfrac{1}{2})\pi\hbar$$

QM1-3 $V'(x) = \frac{1}{2}m\omega^2 x^2 - E_0 q x = \frac{1}{2}m\omega^2\left(x^2 - \frac{2E_0 q}{m\omega^2}x + \left(\frac{E_0 q}{m\omega^2}\right)^2\right) + \text{const}$

$$= \frac{1}{2}m\omega^2\left(x - \frac{E_0 q}{m\omega^2}\right)^2 + \text{const}$$

$t<0$: $x' = x - \underbrace{\frac{E_0 q}{m\omega^2}}_{x_0}$ $\quad \xi' = \xi - \underbrace{\sqrt{\frac{m\omega}{\hbar}}\frac{E_0 q}{m\omega^2}}_{\xi_0}$

$$\psi(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-(\xi-\xi_0)^2/2} \longrightarrow e^{-\frac{\xi^2 - 2\xi\xi_0 + \xi_0^2}{2}}$$

$t=0$:

$$\int dx\, \psi(x)\psi_n(x) = \sqrt{\frac{m\omega}{\pi\hbar}}\frac{1}{\sqrt{2^n n!}}\int H_n(\xi) e^{-\xi^2} e^{\xi\xi_0 - \xi_0^2/2}\, dx$$

$2z = \xi_0$ $\qquad e^{-z^2} = e^{-\xi_0^2/4}$

$$= \sqrt{\frac{m\omega}{\pi\hbar}}\frac{1}{\sqrt{2^n n!}}\int H_n(\xi) e^{-\xi^2} e^{2z\xi - z^2} e^{-\xi_0^2/4}\, dx$$

$$= \sqrt{\frac{m\omega}{\pi\hbar}}\frac{e^{-\xi_0^2/4}}{\sqrt{2^n n!}}\int H_n(\xi)\sum_{k=0}^{\infty}\frac{z^k}{k!}H_k(\xi)\, e^{-\xi^2}\sqrt{\frac{\hbar}{m\omega}}\, d\xi$$

$$= \frac{e^{-\xi_0^2/4}}{\sqrt{2^n n!\,\pi}}\sum_{k=0}^{\infty}\frac{(\xi_0/2)^k}{k!}\underbrace{\int H_n(\xi)H_k(\xi)e^{-\xi^2}\, d\xi}_{\sqrt{\pi}\,2^n n!\,\delta_{nk}}$$

note $\int |\psi_n(x)|^2 dx = \frac{1}{\sqrt{\pi}\,2^n n!}\underbrace{\int H_n(\xi)H_n(\xi)e^{-\xi^2}\, d\xi}_{} = 1$

$$\Rightarrow \quad = \sqrt{\pi}\,2^n n!$$

2013 $$P_n = \left| e^{-\xi_0^2/4} \sqrt{2^n n!}\, \frac{(\xi_0/2)^n}{n!} \right|^2, \quad \xi_0 = \sqrt{\frac{m\omega}{\hbar}}\, \frac{E_0 q}{m\omega^2}$$

QMI-4 must have $\psi\big|_{z=0} = 0$ $\quad (z \propto \cos\theta)$

$P_l^m(-x) = (-1)^{l+m} P_l^m(x) \Rightarrow l+m$ must be odd

$\Rightarrow$ wavefunctions $Y_l^m(\theta,\varphi)$ for $l+m$ odd

$l=1: m=0 \qquad l=3: m=0, \pm 2$

$l=2: m=\pm 1 \qquad l=4: m=\pm 1, \pm 3$

$$H = \frac{\vec{L}^2}{2ma^2}, \quad \text{radius } a$$

$$H\psi = \frac{l(l+1)\hbar^2}{2ma^2}\psi \Rightarrow E_l = \frac{l(l+1)\hbar^2}{2ma^2}, \quad l = 1, 2, \ldots$$

l level is l-degenerate $(m = -l+1, -l+3, \ldots, l-1)$

QMI-5 $H = \omega S_z$, $\quad |\alpha\rangle = a|+\rangle + b|-\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$, $S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$$\langle S_x \rangle_0 = \frac{\hbar}{2}(a\ b)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar}{2}(a\ b)\begin{pmatrix} b \\ a \end{pmatrix} = \frac{\hbar}{2}(2ab) = \hbar ab$$

$$\langle S_y \rangle_0 = \frac{\hbar}{2}(a\ b)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar}{2}(a\ b)\begin{pmatrix} -ib \\ ia \end{pmatrix} = \frac{\hbar}{2}i(ab - ba) = 0$$

$$e^{iHt/\hbar} = \begin{pmatrix} e^{i\omega/2\, t} & \\ & e^{-i\omega/2\, t} \end{pmatrix}$$

$$S_x\big|_t = \frac{\hbar}{2}\begin{pmatrix} e^{i\omega t/2} & \\ & e^{-i\omega t/2} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} e^{-i\omega t/2} & \\ & e^{i\omega t/2} \end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix} e^{i\omega t/2} & \\ & e^{-i\omega t/2} \end{pmatrix}\begin{pmatrix} 0 & e^{i\omega t/2} \\ e^{-i\omega t/2} & 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 & e^{i\omega t} \\ e^{-i\omega t} & 0 \end{pmatrix}$$

$$= S_x\big|_0 \cos\omega t - S_y\big|_0 \sin\omega t$$

$$S_y\big|_t = \frac{\hbar}{2}\begin{pmatrix} e^{i\omega t/2} & \\ & e^{-i\omega t/2} \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} e^{-i\omega t/2} & \\ & e^{i\omega t/2} \end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix} e^{i\omega t/2} & \\ & e^{-i\omega t/2} \end{pmatrix}\begin{pmatrix} 0 & -ie^{i\omega t/2} \\ ie^{-i\omega t/2} & 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 & -ie^{i\omega t} \\ ie^{-i\omega t} & 0 \end{pmatrix}$$

$= S_y\big|_0 \cos\omega t + S_x\big|_0 \sin\omega t \quad \Rightarrow$ same for exp values

$$g \doteq \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \qquad A^\mu = (\Phi, \vec{A}) \qquad j^\mu = (\rho, \vec{J})$$

EM11-1

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \Rightarrow \partial_\mu F^{\mu\nu} = \partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu$$

$$F^{i0} = \partial^i \Phi - \partial^0 A^i = -\nabla\phi - \partial\vec{A}/\partial t = E^i \rightarrow F^{0i} = -E^i$$

$$\partial_\mu F^{\mu 0} = -\partial_i E^i = \nabla\cdot\vec{E}$$

$$\partial_i F^{ij} = \partial_i \partial^i A^j - \partial^j \partial_i A^i = -\nabla^2 A^j - \partial^j \nabla\cdot\vec{A} = -(\nabla^2\vec{A} - \nabla(\nabla\cdot\vec{A}))_j$$

$$= (\nabla\times(\nabla\times\vec{A}))_j = (\nabla\times\vec{B})_j \qquad , \quad \partial_0 F^{0j} = -\partial_0 E^j$$

so $\partial_\mu F^{\mu\nu} = \mu_0 j^\nu \iff \nabla\cdot\vec{E} = \frac{\rho}{\epsilon_0}, \quad \nabla\times\vec{B} - \partial\vec{E}/\partial t = \mu_0\vec{J}$

dual tensor $\mathcal{F}^{\alpha\beta}$: $\vec{E}\to\vec{B}$, $\vec{B}\to-\vec{E}$

then $\partial_\mu \mathcal{F}^{\mu\nu} = 0 \iff \nabla\cdot\vec{B} = 0, \quad \nabla\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0$

Equations are valid in any frame (with F, $\mathcal{F}$ transformed by Lorentz transformation) ✓

$\vec{E}\cdot\vec{B} = 0 \qquad E^i = F^{i0}, \quad B^i = \mathcal{F}^{i0}$

$$\begin{pmatrix} \gamma & \gamma\beta & & \\ \gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \begin{pmatrix} \gamma & \gamma\beta & & \\ \gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} \cdots \end{pmatrix} \begin{pmatrix} -\gamma\beta E_x & -\gamma E_x & -E_y & -E_z \\ \gamma E_x & \gamma\beta E_x & -B_z & B_y \\ \gamma E_y + \gamma\beta B_z & \gamma\beta E_y + \gamma B_z & 0 & -B_x \\ \gamma E_z - \gamma\beta B_y & \gamma\beta E_z - \gamma B_y & B_x & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & & - \cdots & \\ \gamma^2(1-\beta^2)E_x & 0 & & \\ \gamma E_y + \gamma\beta B_z & \gamma\beta E_y + \gamma B_z & 0 & \\ \gamma E_z - \gamma\beta B_y & \gamma\beta E_z - \gamma B_y & B_x & 0 \end{pmatrix}$$

$$E_x' = E_x, \quad E_y' = \gamma(E_y + \beta B_z), \quad E_z' = \gamma(E_z - \beta B_y)$$

$$B_x' = B_x, \quad B_y' = \gamma(B_y - \beta E_z), \quad B_z' = \gamma(B_z + \beta E_y)$$

$$\vec{E}'\cdot\vec{B}' = E_x B_x + \gamma^2(E_y B_y - \beta^2 E_z B_z + \beta(\cancel{B_z B_y} - \cancel{E_z E_y}))$$

$$+ \gamma^2(E_z B_z - \beta^2 E_y B_y + \beta(\cancel{E_y E_z} - \cancel{B_y B_z}))$$

$$= E_x B_x + E_y B_y + E_z B_z = \vec{E}\cdot\vec{B}$$

pure electric: $\vec{B} = 0 \Rightarrow \vec{E}' = (E_x, \gamma E_y, \gamma E_z)$

so $\vec{E}' = 0 \Rightarrow \vec{E} = 0 \quad (\Rightarrow \vec{B}' = 0)$ since $\gamma \neq 0$

$-E_z B_z + E_y B_y + E_x B_x = 0$

2013 EMII-2 $\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = i\omega\vec{B}$, $\nabla\times\vec{B} = \mu\epsilon\frac{\partial\vec{E}}{\partial t} = -i\omega\mu\epsilon\vec{E}$, $\nabla\cdot\vec{E} = \nabla\cdot\vec{B} = 0$

$$\nabla\times(\nabla\times\vec{E}) = i\omega\nabla\times\vec{B} = \mu\epsilon\omega^2\vec{E}$$
$$= \nabla(\overset{0}{\cancel{\nabla\cdot\vec{E}}}) - \nabla^2\vec{E} \quad\Rightarrow\quad (\nabla^2 + \mu\epsilon\omega^2)\vec{E} = 0$$

$$\nabla\times(\nabla\times\vec{B}) = -i\omega\mu\epsilon\nabla\times\vec{E} = \omega^2\mu\epsilon\vec{B}$$
$$= \nabla(\overset{0}{\cancel{\nabla\cdot\vec{B}}}) - \nabla^2\vec{B} \quad\Rightarrow\quad (\nabla^2 + \mu\epsilon\omega^2)\vec{B} = 0$$

$$\vec{E} = \vec{E}_t(x,y)f(z)$$

$$\nabla^2\vec{E} = f\nabla_t^2\vec{E}_t + \vec{E}_t\frac{\partial^2 f}{\partial z^2} = -\mu\epsilon\omega^2\vec{E}_t f$$
$$\nabla_t^2\vec{E}_t + \vec{E}_t\Big(\underbrace{\frac{1}{f}\frac{\partial^2 f}{\partial z^2} + \mu\epsilon\omega^2}_{\text{function of } z\text{, must be constant}}\Big) = 0$$

$$\Rightarrow \frac{\partial^2 f}{\partial z^2} = (\underbrace{-\mu\epsilon\omega^2 + \text{const}}_{\text{call this } -k^2})f$$

$$\Rightarrow \frac{\partial^2}{\partial z^2}\vec{E} = -k^2\vec{E}$$ (must be $-$ so doesn't go to ∞)

so $(\nabla_t^2 + \mu\epsilon\omega^2 - k^2)\vec{E} = 0$, likewise for $\vec{B}$

$\nabla\cdot\vec{E} = \boxed{\nabla_t\cdot\vec{E}_t + \partial_z E_z = 0 \quad \text{likewise } B}$

$\nabla\times\vec{E} = i\omega\vec{B} \Rightarrow \boxed{\hat{z}\cdot\nabla_t\times\vec{E}_t = i\omega B_z}$

$$\hat{z}\times(\nabla\times\vec{E}) = \nabla(\hat{z}\cdot\vec{E}) - (\hat{z}\cdot\nabla)\vec{E} = \nabla E_z - \frac{\partial}{\partial z}(\vec{E}_t + \cancel{E_z\hat{z}})$$
$$= \nabla_t E_z - \frac{\partial\vec{E}_t}{\partial z} \qquad \cancel{\frac{\partial E_z}{\partial z}\hat{z}} + \nabla_t E_z$$

$$\Rightarrow \boxed{\nabla_t E_z - \frac{\partial\vec{E}_t}{\partial z} = i\omega\,\hat{z}\times\vec{B}_t}$$

$\nabla\times\vec{B} = -i\mu\epsilon\omega\vec{E} \Rightarrow \boxed{\hat{z}\cdot\nabla_t\times\vec{B}_t = -i\mu\epsilon\omega E_z}$

$$\boxed{\nabla_t B_z - \underset{\uparrow\, ik\vec{B}_t}{\frac{\partial\vec{B}_t}{\partial z}} = -i\mu\epsilon\omega\,\hat{z}\times\vec{E}_t}$$

$$\boxed{\begin{aligned}\nabla_t E_z &= ik\vec{E}_t + i\omega\hat{z}\times\vec{B}_t\\ \nabla_t B_z &= ik\vec{B}_t - i\mu\epsilon\omega\hat{z}\times\vec{E}_t\end{aligned}}\quad \begin{aligned}&\Rightarrow \vec{E}_t = \frac{1}{ik}\left(\nabla_t E_z - i\omega\hat{z}\times\vec{B}_t\right)\\ &\Rightarrow \vec{B}_t = \frac{1}{ik}\left(\nabla_t B_z + i\mu\epsilon\omega\,\hat{z}\times\vec{E}_t\right)\end{aligned}$$

$$\vec{E}_t = \frac{1}{ik}\nabla_t E_z - \frac{\omega}{ik^2}\left(\hat{z}\times\nabla_t B_z - i\mu\epsilon\omega\,\vec{E}_t\right)$$

$$\vec{E}_t = \frac{1}{ik}\nabla_t E_z - \frac{\omega}{ik}\hat{z}\times\nabla_t B_z + \mu\epsilon\frac{\omega^2}{k^2}\vec{E}_t$$

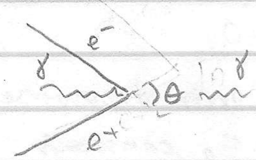
$$\Rightarrow \vec{E}_t = \frac{\nabla_t E_z - \omega\hat{z}\times\nabla_t B_z}{ik\left(1-\mu\epsilon\,\omega^2/k^2\right)} = \frac{i}{\mu\epsilon\omega^2-k^2}\left(k\nabla_t E_z - \omega\hat{z}\times\nabla_t B_z\right)$$

$$\Rightarrow \vec{B}_t = \frac{1}{ik}\left(\nabla_t B_z + i\mu\epsilon\omega\frac{i}{\mu\epsilon\omega^2-k^2}\left(k\hat{z}\times\nabla_t E_z + \omega\nabla_t B_z\right)\right)$$

$$= \frac{i}{\mu\epsilon\omega^2-k^2}\left(\frac{\mu\epsilon\omega^2-k^2}{-k}\nabla_t B_z + \mu\epsilon\omega\,\hat{z}\times\nabla_t E_z + \cancel{\frac{\mu\epsilon\omega^2}{k}\nabla_t B_z}\right)$$

$$= \frac{i}{\mu\epsilon\omega^2-k^2}\left(k\nabla_t B_z + \mu\epsilon\omega\,\hat{z}\times\nabla_t E_z\right)$$

EM11-3 lab frame



$E_e = \gamma m_e \;,\; \gamma^2 m_e^2 = p_e^2 + m_e^2$

$p_e = m_e\sqrt{\gamma^2-1}$ $\qquad\left(\text{note } \sqrt{\gamma^2-1} = \sqrt{\frac{1-1+\beta^2}{1-\beta^2}} = \gamma v\right)$

$p_{ex} = p_e\cos(\theta/2)$ $\qquad(\gamma: E=p)$

$$\begin{aligned}E_1 - E_2 &= 2m_e\sqrt{\gamma^2-1}\cos(\theta/2)\\ E_1+E_2 &= 2\gamma m_e\end{aligned} \Rightarrow \begin{aligned}E_1 &= m_e\left(\gamma+\sqrt{\gamma^2-1}\cos(\theta/2)\right)\\ E_2 &= m_e\left(\gamma-\sqrt{\gamma^2-1}\cos(\theta/2)\right)\end{aligned}$$

$$\gamma\gg 1:\ \sqrt{\gamma^2-1} = \gamma\left(1-\gamma^{-2}\right)^{1/2} \simeq \gamma\left(1-\tfrac{1}{2}\gamma^{-2}\right) = \gamma - \tfrac{1}{2}\gamma^{-1}$$

$$\theta\to 0 \Rightarrow E_1 \simeq 2\gamma m_e - \frac{m_e}{2\gamma}$$

$$E_2 \simeq \frac{m_e}{2\gamma}$$

2013 EMII-4

$\nabla\cdot\vec{P}=0, \quad \vec{D}=\epsilon_0\vec{E}+\vec{P}=\epsilon_0\vec{E}+\gamma\nabla\times\vec{E}$

$\Rightarrow \nabla\cdot\vec{D}=\nabla\cdot(\epsilon_0\vec{E}+\gamma\nabla\times\vec{E})=\epsilon_0\nabla\cdot\vec{E}=0 \Rightarrow \nabla\cdot\vec{E}=0$

$\nabla\times\vec{H}=\frac{1}{\mu_0}\nabla\times\vec{B}=\frac{\partial\vec{D}}{\partial t}=\epsilon_0\frac{\partial\vec{E}}{\partial t}+\gamma\nabla\times\left(\frac{\partial\vec{E}}{\partial t}\right)$

$\nabla\times\vec{E}=\frac{-\partial\vec{B}}{\partial t} \Rightarrow \underbrace{\nabla\times(\nabla\times\vec{E})}_{-\nabla^2\vec{E}}=-\frac{\partial}{\partial t}\left(\mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}+\mu_0\gamma\nabla\times\frac{\partial\vec{E}}{\partial t}\right)$

$\nabla^2\vec{E}=\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(\vec{E}+\frac{\gamma}{\epsilon_0}\nabla\times\vec{E}\right)$

$\vec{E}=\vec{E}_0 e^{ikz-i\omega t}, \quad \nabla\times\vec{E}=\hat{x}(\partial_y E_z-ikE_y)+\hat{y}(ikE_x-\partial_x E_z)+\hat{z}(\partial_x E_y-\partial_y E_x)$

$(-k^2+\nabla_t^2)\vec{E}_0=\frac{-\omega^2}{c^2}\left(\vec{E}_0+\frac{\gamma}{\epsilon_0}(ik\hat{z}\times\vec{E}_0+\nabla\times\vec{E}_0)\right)$

transverse modes all vectors point transversely except $\nabla\times\vec{E}_0$,

so $\nabla\times\vec{E}_0=0$

$\left(\nabla_t^2+\frac{\omega^2}{c^2}-k^2\right)\vec{E}_0=\frac{-\omega^2\gamma}{c^2\epsilon_0}ik\,\hat{z}\times\vec{E}_0$

longitudinal modes: $\vec{E}_0=E_0\hat{z}$; $\hat{z}\times\vec{E}_0=0$,

$\nabla\times\vec{E}_0=\hat{x}\partial_y E_0-\hat{y}\partial_x E_0=0$, only transverse terms

$\Rightarrow \partial_y E_0=\partial_x E_0=0 \rightarrow E_0(x,y)=E_0$ constant

$\Rightarrow \left(\frac{\omega^2}{c^2}-k^2\right)E_0=0$

$\Rightarrow$ only longitudinal modes are pure plane waves (no transverse variation) with $\omega=ck$

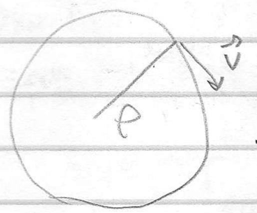
EM11-5 $ds^2 = -dt^2 + dr^2$, $d\tau^2 = -ds^2 = dt^2 - dr^2$ $(dr^2 = dx^2 + dy^2 + dz^2)$

$$d\tau^2 = dt^2 - \frac{dr^2}{dt^2}dt^2 = dt^2(1-\beta^2) \Rightarrow d\tau = dt\sqrt{1-\beta^2} = \frac{dt}{\gamma}$$

$$v^\alpha = \frac{dx^\alpha}{d\tau} = \left(\frac{dt}{d\tau}, \frac{d\vec r}{d\tau}\right), \text{ but } \frac{dt}{d\tau} = \gamma, \ \frac{d\vec r}{d\tau} = \frac{d\vec r}{dt}\frac{dt}{d\tau} = \gamma\vec v$$

$$= (\gamma, \gamma\vec v)$$

$$\eta = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$



$$\frac{d\vec v}{dt} = \frac{v^2}{\rho}. \quad P = \frac{2e^2}{3c^3}\frac{d\vec v}{dt}\cdot\frac{d\vec v}{dt}$$

$$\rightarrow P = \frac{-2e^2}{3c^2}\frac{dv^\alpha}{d\tau}\frac{dv_\alpha}{d\tau}$$

since $-\frac{dv^\alpha}{d\tau}\frac{dv_\alpha}{d\tau} = \left|\frac{d(\gamma\vec v)}{d\tau}\right|^2 - \left|\frac{d\gamma}{d\tau}\right|^2$, $\frac{d\gamma}{dv} = \frac{-v}{\gamma^3}$

$$= \left|\frac{d(\gamma\vec v)}{d\tau}\right|^2 - \underbrace{\left|\frac{v}{\gamma^3}\frac{dv}{d\tau}\right|^2}_{\propto v^2 \to 0 \text{ for } v \ll c}$$

$$\frac{d(\gamma\vec v)}{d\tau} = \vec v v\gamma^3\frac{dv}{d\tau} + \gamma\frac{d\vec v}{d\tau}$$

but $\frac{dv}{d\tau} = 0$ for circular motion

$$\Rightarrow P = \frac{2e^2}{3c^3}\left|\gamma\frac{d\vec v}{d\tau}\right|^2, \quad \frac{d\vec v}{d\tau} = \frac{d\vec v}{dt}\frac{dt}{d\tau} = \gamma\frac{d\vec v}{dt}$$

$$\Rightarrow P = \frac{2e^2}{3c^3}\gamma^4\left|\frac{dv}{dt}\right|^2 = \frac{2e^2}{3c^3}\gamma^4\frac{v^4}{\rho^2} = \frac{2e^2c}{3\rho^2}\beta^4\gamma^4$$

General: $P = \frac{2e^2}{3c^3}\left(\left|\underbrace{\frac{d(\gamma\vec v)}{dt}}_{\vec\alpha}\right|^2 - \left|\underbrace{\frac{d\gamma}{dt}}_{\alpha_0}\right|^2\right)$

$$\vec\alpha = \vec v v\gamma^4\frac{dv}{dt} + \gamma^2\frac{d\vec v}{dt} = \gamma^4\vec\beta\beta\dot{\vec v} + \gamma^2\dot{\vec v}$$

$$\alpha_0 = -v\gamma^3\frac{dv}{d\tau} = -\gamma^4\beta\dot v$$

2013 QMII-1 $H = \frac{p^2}{2m} + k_1 x^2 + k_2 y^2 \quad , \quad L_z = x p_y - y p_x$

$$[p^2, L_z] = [p_x^2, x] p_y - [p_y^2, y] p_x = -i\hbar(2p_x p_y - 2p_y p_x) = 0$$
$$[x^2, L_z] = -y[x^2, p_x] = -y\, i\hbar 2x$$
$$[y^2, L_z] = x[y^2, p_y] = x\, i\hbar 2y$$
$$[H, L_z] = i\hbar\left(-2xyk_1 + 2xyk_2\right) = 2i\hbar xy(k_2 - k_1)$$

L_z conserved $\Longleftrightarrow k_1 = k_2$

QMII-2 $c_1(0) = 1, \; c_2(0) = 0 \qquad V_{12} = \gamma e^{i\omega t}, \; V_{21} = \gamma e^{-i\omega t} \qquad \omega_{12} = -\omega_{21} = \frac{E_1 - E_2}{\hbar}$

$$i\hbar \frac{dc_1}{dt} = V_{12} e^{i\omega_{12} t} c_2 = \gamma e^{i(\omega - \omega_{21})t} c_2$$
$$i\hbar \frac{dc_2}{dt} = \gamma e^{i(-\omega + \omega_{21})t} c_1$$
$$\frac{dc_1}{dt} = \frac{\gamma}{i\hbar} e^{i(\omega - \omega_{21})t} c_2 \quad , \quad \frac{dc_2}{dt} = \frac{\gamma}{i\hbar} e^{-i(\omega - \omega_{21})t} c_1$$
$$\frac{d^2 c_1}{dt^2} = \frac{\gamma}{\hbar}(\omega - \omega_{21}) \frac{i\hbar}{\gamma} \frac{dc_1}{dt} + \frac{\gamma}{i\hbar} e^{i(\omega - \omega_{12})t} \frac{dc_2}{dt}$$
$$\frac{d^2 c_1}{dt^2} = i(\omega - \omega_{21}) \frac{dc_1}{dt} - \left(\frac{\gamma}{\hbar}\right)^2 c_1$$

try $c_1 \propto e^{ikt} \Rightarrow -k^2 = -k(\omega - \omega_{21}) - \left(\frac{\gamma}{\hbar}\right)^2$

$$k^2 - (\omega - \omega_{21})k - \gamma^2/\hbar^2 = 0 \Rightarrow k = \frac{\omega - \omega_{21}}{2} \pm \sqrt{\left(\frac{\omega - \omega_{21}}{2}\right)^2 + \frac{\gamma^2}{\hbar^2}}$$
$$\Rightarrow c_1 \propto e^{i\left(\frac{\omega - \omega_{21}}{2}\right)t} e^{\pm i\sqrt{\left(\frac{\omega - \omega_{21}}{2}\right)^2 + \gamma^2/\hbar^2}\, t}$$
$$c_1 = A e^{i\frac{\omega - \omega_{21}}{2}t} e^{i\sqrt{\left(\frac{\omega - \omega_{21}}{2}\right)^2 + \gamma^2/\hbar^2}\, t} + B e^{i\frac{\omega - \omega_{21}}{2}t} e^{-i\sqrt{\left(\frac{\omega - \omega_{21}}{2}\right)^2 + \gamma^2/\hbar^2}\, t}$$
$$c_1(0) = A + B = 1$$
$$c_2(0) = 0 \Rightarrow 0 = \dot{c}_1(0) = i\left[\frac{\omega - \omega_{21}}{2} + \sqrt{\left(\frac{\omega - \omega_{21}}{2}\right)^2 + \gamma^2/\hbar^2}\right] A + i\left[\frac{\omega - \omega_{21}}{2} - \sqrt{\left(\frac{\omega - \omega_{21}}{2}\right)^2 + \gamma^2/\hbar^2}\right] B$$
$$\left[\frac{\omega - \omega_{21}}{2} + \sqrt{\left(\frac{\omega - \omega_{21}}{2}\right)^2 + \gamma^2/\hbar^2}\right] A + \left[\frac{\omega - \omega_{21}}{2} - \sqrt{\left(\frac{\omega - \omega_{21}}{2}\right)^2 + \gamma^2/\hbar^2}\right](1 - A) = 0$$
$$2\sqrt{\left(\frac{\omega - \omega_{21}}{2}\right)^2 + \gamma^2/\hbar^2}\, A + \frac{\omega - \omega_{21}}{2} - \sqrt{\left(\frac{\omega - \omega_{21}}{2}\right)^2 + \gamma^2/\hbar^2} = 0$$
$$\Rightarrow A = \frac{1}{2}\left[\frac{-\frac{\omega - \omega_{21}}{2}}{\sqrt{\left(\frac{\omega - \omega_{21}}{2}\right)^2 + \gamma^2/\hbar^2}} + 1\right]$$
$$B = \frac{1}{2}\left[\frac{\frac{\omega - \omega_{21}}{2}}{\sqrt{\left(\frac{\omega - \omega_{21}}{2}\right)^2 + \gamma^2/\hbar^2}} + 1\right]$$

TDPT:

$$c_2 \simeq -\frac{i}{\hbar}\int_0^t dt'\, e^{i\omega_{21}t'}\,\gamma e^{-i\omega t'}\,dt' = -\frac{i\gamma}{\hbar}\int_0^t dt'\, e^{-i(\omega-\omega_{21})t'}\,dt'$$

$$\simeq -\frac{i\gamma/\hbar}{-i(\omega-\omega_{12})}\left(e^{-i(\omega-\omega_{21})t} - 1\right)$$

$$\simeq \frac{\gamma}{(\omega-\omega_{21})\hbar}\left(e^{-i(\omega-\omega_{21})t} - 1\right)$$

$$|c_2(t)|^2 \simeq \frac{\gamma^2}{(\omega-\omega_{21})^2\hbar^2}\left(2 - 2\cos\left((\omega-\omega_{21})t\right)\right) = \frac{4\gamma^2}{(\omega-\omega_{21})^2\hbar^2}\sin^2\left(\frac{\omega-\omega_{21}}{2}t\right)$$

$$|c_1(t)|^2 \simeq 1 - \frac{4\gamma^2}{(\omega-\omega_{21})^2\hbar^2}\sin^2\left(\frac{\omega-\omega_{21}}{2}t\right)$$

Exact:

$$c_1 = e^{i\frac{\omega-\omega_{21}}{2}}\left[(A+B)\cos\left(\sqrt{\left(\frac{\omega-\omega_{21}}{2}\right)^2 + \frac{\gamma^2}{\hbar^2}}\,t\right) + (A-B)i\sin\left(\text{same}\right)\right]$$

$$= e^{i\frac{\omega-\omega_{21}}{2}}\left[\cos\left(\sqrt{\left(\frac{\omega-\omega_{21}}{2}\right)^2+\frac{\gamma^2}{\hbar^2}}\,t\right) - \frac{i}{\sqrt{1+\frac{4\gamma^2}{\hbar^2(\omega-\omega_{21})^2}}}\sin\left(\sqrt{\left(\frac{\omega-\omega_{21}}{2}\right)^2+\gamma^2/\hbar^2}\,t\right)\right]$$

$$|c_1(t)|^2 = \cos^2\left(\sqrt{\left(\frac{\omega-\omega_{21}}{2}\right)^2+\frac{\gamma^2}{\hbar^2}}\,t\right) + \left(1+\frac{4\gamma^2}{\hbar^2(\omega-\omega_{21})^2}\right)^{-1}\sin^2\left(\sqrt{\left(\frac{\omega-\omega_{21}}{2}\right)^2+\gamma^2/\hbar^2}\,t\right)$$

and $|c_2(t)|^2 = 1 - |c_1(t)|^2 = \sin^2(\ldots) - (\ldots)^{-1}\sin^2(\ldots)$ $\quad\left(\omega_{21} = \frac{E_2-E_1}{\hbar}\right)$

γ small: $\left(1+\frac{4\gamma^2}{\hbar^2(\omega-\omega_{21})^2}\right)^{-1} \simeq 1 - \frac{4\gamma^2}{\hbar^2(\omega-\omega_{21})^2}$, $\quad \sqrt{\left(\frac{\omega-\omega_{21}}{2}\right)^2+\gamma^2/\hbar^2} \simeq \frac{\omega-\omega_{21}}{2}$

$$\Rightarrow |c_1|^2 \simeq 1 - \frac{4\gamma^2}{\hbar^2(\omega-\omega_{21})^2}\sin^2\left(\frac{\omega-\omega_{21}}{2}t\right) \qquad (\gamma \ll (\omega-\omega_{21})\hbar)$$

same as PT result

If $\omega-\omega_{21} \ll \frac{\gamma}{\hbar}$ then

$$c_1 \simeq e^{i\frac{\omega-\omega_{21}}{2}}\left[\cos\left(\frac{\gamma}{\hbar}t\right) - \frac{\hbar(\omega-\omega_{21})}{2\gamma}\sin\left(\frac{\gamma}{\hbar}t\right)\right]$$

$$|c_1|^2 \simeq \cos^2\left(\frac{\gamma t}{\hbar}\right) - \frac{\hbar(\omega-\omega_{21})}{\gamma}\sin\left(\frac{\gamma}{\hbar}t\right)\cos\left(\frac{\gamma}{\hbar}t\right)$$

2013 QMII-3 $V(r) = -Ae^{-r/a}$

spherical symmetry $\Rightarrow \psi = \psi(r)$

also drops off very quickly. So try

$$\psi(r) = \sqrt{\frac{3}{4\pi R^3}} \begin{cases} 1, & r<R \\ 0, & r>R \end{cases}$$

$$H = \frac{-\hbar^2}{2mr^2}\frac{d}{dr}r^2\frac{d}{dr} - Ae^{-r/a} \quad ; \quad \psi^* H \psi = \frac{3}{4\pi R^3}\int_0^R (-A)e^{-r/a} 4\pi r^2 dr$$

$$\langle\psi|H|\psi\rangle = \frac{-3A}{R^3}\int_0^R r^2 e^{-r/a} dr = \frac{-3A}{R^3}\left[-aR^2 e^{-R/a} + 2a\int_0^R r e^{-r/a} dr\right]$$

$$= \frac{-3A}{R^3}\left[-aR^2 e^{-R/a} - 2a^2 R e^{-R/a} + 2a^2 \underbrace{\int_0^R e^{-r/a} dr}_{a(1-e^{-R/a})}\right]$$

$$= -\frac{3A}{R^3}\left[2a^3 - (2a^3 + 2a^2 R + aR^2)e^{-R/a}\right]$$

$$= -6A\left[a^3R^{-3} - (a^3R^{-3} + a^2R^{-2} + \frac{a}{2}R^{-1})e^{-R/a}\right] = \bar{H}$$

$$\frac{d}{d\alpha}\bar{H} = -6A\left[-3a^3R^{-4} + (3a^3R^{-4} + 2a^2R^{-3} + \frac{a}{2}R^{-2} + a^2R^{-3} + aR^{-2} + \frac{R^{-1}}{2})e^{-R/a}\right] = 0$$

$$3a^3 e^{R/a} = 3a^3 + 3a^2 R + \frac{3}{2}aR^2 + \frac{1}{2}R^3$$

$$e^{R/a} = 1 + \frac{R}{a} + \frac{1}{2}\left(\frac{R}{a}\right)^2 + \frac{1}{6}\left(\frac{R}{a}\right)^3 \qquad \text{no solution.}$$

Try $\psi(r) = Ce^{-\alpha r/2a}$

$$1 = \langle\psi|\psi\rangle = C^2 4\pi\int_0^\infty r^2 e^{-\frac{\alpha}{a}r} dr = C^2 4\pi \frac{2a^3}{\alpha^3} \Rightarrow C = \sqrt{\frac{\alpha^3}{8\pi a^3}}$$

$$\frac{d\psi}{dr} = -C\frac{\alpha}{2a}e^{-\alpha/2a\, r}$$

$$\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right) = -C\frac{\alpha}{2a}\left(2r - \frac{\alpha r^2}{2a}\right)e^{-\alpha/2a\, r}$$

$$\psi^* H\psi = \left(\frac{\hbar^2}{2m}\frac{\alpha}{2a}\left(\frac{2}{r} - \frac{\alpha}{2a}\right) - Ae^{-r/a}\right)C^2 e^{-\alpha r/a}$$

$$\langle\psi|H|\psi\rangle = C^2\left[\frac{\hbar^2}{2m}\frac{\alpha}{2a}\left(8\pi\underbrace{\int r dr e^{-\frac{\alpha}{a}r}}_{a^2/\alpha^2} - \frac{2\pi\alpha}{a}\underbrace{\int r^2 dr e^{-\frac{\alpha}{a}r}}_{2a^3/\alpha^3}\right) - A4\pi\underbrace{\int r^2 dr e^{-(1+\alpha)\frac{r}{a}}}_{2a^3/(1+\alpha)^3}\right]$$

$$\bar{H} = \frac{\alpha^3}{8\pi a^3}\left(\frac{\hbar^2}{2m}\frac{\alpha}{2a}\left(8\pi\, a^2/\alpha^2 - 4\pi\, a^2/\alpha^2\right) - 8\pi A \frac{a^3}{(1+\alpha)^3}\right)$$

$$= \frac{\hbar^2\alpha^2}{8ma^2} - \frac{A\alpha^3}{(1+\alpha)^3}$$

$$\frac{\partial \bar{H}}{\partial \alpha} = \frac{\hbar^2\alpha}{4ma^2} - A\frac{3\alpha^2(1+\alpha)^3 - \alpha^3 3(1+\alpha)^2}{(1+\alpha)^6} = \frac{\hbar^2\alpha}{4ma^2} - \frac{3A\alpha^2}{(1+\alpha)^4} = 0$$

$$\frac{\hbar^2}{4ma^2}(1+\alpha)^4 = 3A\alpha \qquad A = 32\ \mathrm{MeV} \qquad a = 2.2\,\mathrm{fm}$$

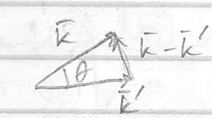
$$(1+\alpha)^4 = \frac{12\,ma^2A}{\hbar^2}\alpha \qquad m = \frac{m_p m_n}{m_p+m_n} \simeq 470\ \mathrm{MeV}/c^2$$

$$(1+\alpha)^4 \simeq 22.4\,\alpha \Rightarrow \alpha \simeq 1.34$$

$$\Rightarrow \bar{H}_0 = A\left(\frac{\hbar^2}{8ma^2A}\alpha^2 - \frac{\alpha^3}{(1+\alpha)^3}\right) \simeq A\left(\frac{3}{2}\frac{1}{22.4}\alpha^2 - \frac{\alpha^3}{(1+\alpha)^3}\right)$$

$$\simeq -3.58\ \mathrm{MeV}$$

QMII-4 $\quad f(\bar{k}', \bar{k}) = \frac{-1}{4\pi}\frac{2m}{\hbar^2}\int d^3\bar{x}\, e^{i(\bar{k}-\bar{k}')\cdot\bar{x}}\, u(\bar{x})$



$$q \equiv |\bar{k}-\bar{k}'| = \sqrt{2k^2 - 2k^2\cos\theta} = k\sqrt{2-2\cos\theta} = 2k\sin(\theta/2)$$

$$(\bar{k}-\bar{k}')\cdot\bar{x} = 2k\sin(\theta/2)\, r\cos\theta', \quad \bar{k}-\bar{k}' \text{ along } z$$

$$\int d\Omega\, e^{i(\bar{k}-\bar{k}')\cdot\bar{x}} = \int_{-1}^{1} 2\pi\, d\mu\, e^{i2k\sin(\theta/2) r\mu} = \frac{2\pi}{iqr}\left(e^{iqr} - e^{-iqr}\right) = \frac{4\pi}{qr}\sin(qr)$$

$$f(\theta) = -\frac{2m}{\hbar^2}\frac{1}{q}\int_0^\infty r\, dr\, \sin(qr)\, U_0 e^{-r^2/R^2}$$

$$= \frac{-2mU_0}{\hbar^2 q}\int_0^\infty dr\, r\sin(qr) e^{-r^2/R^2} = \frac{-mU_0}{i\hbar^2 q}\left[\int_0^\infty dr\, r\, e^{iqr - \frac{r^2}{R^2}} - \int_0^\infty dr\, r\, e^{-iqr-\frac{r^2}{R^2}}\right]$$

$$e^{iqr - \frac{r^2}{R^2}} = e^{-\left(\frac{r}{R} - iqR/2\right)^2} e^{-q^2R^2/4} \qquad e^{-iqr - r^2/R^2} = e^{-\left(\frac{r}{R}+iqR/2\right)^2} e^{-q^2R^2/4}$$

$$u = \frac{r}{R} \mp iqR/2 \quad du = \frac{dr}{R}$$

$$= \frac{-mU_0}{i\hbar^2 q} e^{-q^2R^2/4} R^2 \underbrace{\left[\int_{-iqR/2}^{\infty}(u + iqR/2)\, du\, e^{-u^2} - \int_{iqR/2}^{\infty}(u - iqR/2)\, du\, e^{-u^2}\right]}_{\int_{-iqR/2}^{iqR/2} u\, du\, e^{-u^2} \to 0 \;+\; iqR/2\sqrt{\pi}} \quad r = R(u \pm iqR/2)$$

$$= -\frac{mU_0R^3\sqrt{\pi}}{2\hbar^2} e^{-q^2R^2/4} \Rightarrow \frac{d\sigma}{d\Omega} = |f(\theta)|^2 = u^2R^2\exp\left(-\frac{q^2R^2}{2}\right),$$

$$u = \frac{\sqrt{\pi}\, m U_0 R^2}{2\hbar^2}$$

2013 $\sigma = u^2R^2 \int_{-1}^{1} 2\pi\, d\mu\, e^{-k^2R^2(1-\mu)} = 2\pi u^2R^2 e^{-k^2R^2} \underbrace{\int_{-1}^{1} d\mu\, e^{k^2R^2\mu}}_{\frac{e^{k^2R^2}-e^{-k^2R^2}}{k^2R^2}}$

$\mu = \cos\theta = 1 - 2\sin^2\theta/2 \Rightarrow \sin^2\theta/2 = \frac{1}{2}(1-\mu)$

$$\sigma = \frac{2\pi u^2}{k^2}\left(1 - e^{-2k^2R^2}\right)$$

QMII-5 $++, +-, -+, --$ index order $\quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\chi_0 = \frac{1}{\sqrt{2}}\left(\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\right) \qquad \chi_0 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix} \qquad \chi_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$

$\chi_1 = \frac{1}{\sqrt{2}}\left(\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\right)$

$\chi_2 = \begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix} \qquad \chi_2 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad \chi_3 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

$\chi_3 = \begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}$

$$S^2_{total} = (\vec{S}_1 + \vec{S}_2)^2 = \frac{\hbar^2}{4}\left((\sigma_1+\sigma_1)^2 + (\sigma_2+\sigma_2)^2 + (\sigma_3+\sigma_3)^2\right)$$

$$= \frac{\hbar^2}{4}\left[\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}^2 + \begin{pmatrix} 0 & -i & -i & 0 \\ i & 0 & 0 & -i \\ i & 0 & 0 & -i \\ 0 & i & i & 0 \end{pmatrix}^2 + \begin{pmatrix} 2 & & & \\ & 0 & & \\ & & 0 & \\ & & & 2 \end{pmatrix}^2\right]$$

$$= \frac{\hbar^2}{4}\left[\begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 0 & -2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ -2 & 0 & 0 & 2 \end{pmatrix} + \begin{pmatrix} 4 & & & \\ & 0 & & \\ & & 0 & \\ & & & 4 \end{pmatrix}\right]$$

$$= \frac{\hbar^2}{4}\begin{pmatrix} 8 & 0 & 0 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 0 & 0 & 8 \end{pmatrix} = \hbar^2\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

eigenvalues, $\hbar^2$:
$(2-\lambda)^2((1-\lambda)^2-1) = (\lambda-2)^3\lambda = 0$
$\Rightarrow \lambda = 0, \ \lambda = 2$

$S^2_{total}\chi_0 = \hbar^2 0 = 0$ (singlet)

$S^2_{total}\chi_1 = \hbar^2 2\chi_1 = 2\hbar^2\psi_1$

$S^2_{total}\chi_2 = 2\hbar^2\chi_2$ (triplet)

$S^2_{total}\chi_3 = 2\hbar^2\chi_3$

para: singlet χ_0 (antisym.) $\Rightarrow \psi_0(\vec r_1, \vec r_2)$ symmetric

ortho: χ_1 (symmetric) $\Rightarrow \psi_1(\vec r_1, \vec r_2)$ antisymmetric

(total wavefunction antisymmetric under exchange)

antisymmetric $\psi \Rightarrow e^-$ farther apart because

$$\psi\Big|_{\vec r_1 - \vec r_2 = 0} = \psi(\vec r_1, \vec r_1) = -\psi(\vec r_1, \vec r_1) = 0$$

$\Rightarrow$ lower energy (weaker e-e interaction) for ortho state

2012 CM1 use $x_\pm = \frac{1}{2}(x \pm y)$, note $x = x_+ + x_-$, $y = x_+ - x_-$

$$x^2 + y^2 = 2(x_+^2 + x_-^2), \quad xy = x_+^2 - x_-^2$$

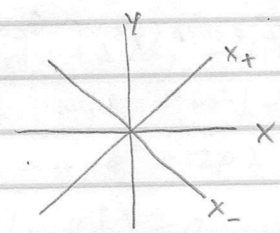
$$L = \frac{1}{2}m\left(2\dot x_+^2 + 2\dot x_-^2 + 2\alpha \dot x_+^2 - 2\alpha \dot x_-^2\right) - \frac{1}{2}k\left(2x_+^2 + 2x_-^2 + 2\beta x_+ - 2\beta x_-\right)$$

$$= m(1+\alpha)\dot x_+^2 + m(1-\alpha)\dot x_-^2 - k(1+\beta)x_+^2 - k(1-\beta)x_-^2$$

independent oscillators: $\omega_+ = \sqrt{\frac{k}{m}\frac{1+\beta}{1+\alpha}}$, $\omega_- = \sqrt{\frac{k}{m}\frac{1-\beta}{1-\alpha}}$

stable oscillation: ω_- real $\Leftrightarrow$ ($\alpha < 1$ and $\beta < 1$) or ($\alpha > 1$ and $\beta > 1$)

eigenvectors: x_+ and x_- ($x+y$ and $x-y$)



Relative frequencies of x_+, x_- vary with α, β but the eigenmodes themselves do not.

Alternatively: $m\ddot x + m\alpha \ddot y + kx + k\beta y = 0$
$m\ddot y + m\alpha \ddot x + ky + k\beta x = 0$ $\Rightarrow T = \begin{pmatrix} m & m\alpha \\ m\alpha & m \end{pmatrix}$ $V = \begin{pmatrix} k & k\beta \\ k\beta & k \end{pmatrix}$

$$T\ddot{\vec x} + V\vec x = 0$$

$$(-\omega^2 T + V)\vec x = 0 \Rightarrow$$

$$\det\begin{pmatrix} -\omega^2 m + k & -\omega^2 m\alpha + k\beta \\ -\omega^2 m\alpha + k\beta & -\omega^2 m + k \end{pmatrix} = (k - \omega^2 m)^2 - (k\beta - \omega^2 m\alpha)^2 = 0$$

$$= \omega^4 m^2(1-\alpha^2) - 2\omega^2 km(1-\beta\alpha) + k^2(1-\beta^2)$$

$$\Rightarrow \omega^2 = \frac{km(1-\beta\alpha) \pm \sqrt{k^2m^2(1-\beta\alpha)^2 - m^2(1-\alpha^2)k^2(1-\beta^2)}}{m^2(1-\alpha^2)} = \frac{k}{m}\frac{1-\beta\alpha \pm \sqrt{1-2\beta\alpha+\beta^2\alpha^2 - \cancel{1} + \alpha^2 + \beta^2 - \cancel{\alpha^2\beta^2}}}{1-\alpha^2}$$

$$= \frac{k}{m}\frac{1-\beta\alpha \pm \alpha \mp \beta}{(1-\alpha)(1+\alpha)} = \frac{k}{m}\frac{(1\mp\beta)(1\pm\alpha)}{(1-\alpha)(1+\alpha)} = \frac{k}{m}\frac{1\mp\beta}{1\mp\alpha}$$

$$-\omega^2 T + V = k\begin{pmatrix} 1 - \frac{1\mp\beta}{1\mp\alpha} & \beta - \frac{1\mp\beta}{1\mp\alpha}\alpha \\ \beta - \frac{1\mp\beta}{1\mp\alpha}\alpha & 1 - \frac{1\mp\beta}{1\mp\alpha} \end{pmatrix} = \frac{k}{1\mp\alpha}\begin{pmatrix} \mp\alpha \pm \beta & \beta(1\mp\alpha) - \alpha(1\mp\beta) \\ \beta - \alpha & \pm(\beta - \alpha) \end{pmatrix} = k\frac{\beta-\alpha}{1\mp\alpha}\begin{pmatrix} \pm 1 & 1 \\ 1 & \pm 1 \end{pmatrix}$$

$\Rightarrow$ vectors $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$