para: singlet $\underset{}{\chi_0}$ (antisym.) $\Rightarrow \psi_0(\vec{r}_1, \vec{r}_2)$ symmetric

ortho: χ_1 (symmetric) $\Rightarrow \psi_1(\vec{r}_1, \vec{r}_2)$ antisymmetric

(total wavefunction antisymmetric under exchange)

antisymmetric $\psi \Rightarrow e^-$ farther apart because

$$\psi\big|_{\vec{r}_1 - \vec{r}_2 = 0} = \psi(\vec{r}_1, \vec{r}_1) = -\psi(\vec{r}_1, \vec{r}_1) = 0$$

$\Rightarrow$ lower energy (weaker e-e interaction) for ortho state

2012 CM1 use $x_\pm = \frac{1}{2}(x \pm y)$, note $x = x_+ + x_-$, $y = x_+ - x_-$

$$x^2 + y^2 = 2(x_+^2 + x_-^2), \quad xy = x_+^2 - x_-^2$$

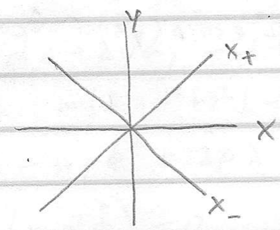
$$L = \frac{1}{2}m\left(2\dot{x}_+^2 + 2\dot{x}_-^2 + 2\alpha\dot{x}_+^2 - 2\alpha\dot{x}_-^2\right) - \frac{1}{2}k\left(2x_+^2 + 2x_-^2 + 2\beta x_+ - 2\beta x_-\right)$$

$$= m(1+\alpha)\dot{x}_+^2 + m(1-\alpha)\dot{x}_-^2 - k(1+\beta)x_+^2 - k(1-\beta)x_-^2$$

independent oscillators: $\omega_+ = \sqrt{\frac{k}{m}\frac{1+\beta}{1+\alpha}}$, $\omega_- = \sqrt{\frac{k}{m}\frac{1-\beta}{1-\alpha}}$

stable oscillation: ω_- real $\Leftrightarrow (\alpha < 1$ and $\beta < 1)$ or $(\alpha > 1$ and $\beta > 1)$

eigenvectors: x_+ and x_- $(x+y$ and $x-y)$



Relative frequencies of x_+, x_- vary with α, β but the eigenmodes themselves do not.

Alternatively: $m\ddot{x} + m\alpha\ddot{y} + kx + k\beta y = 0$
$m\ddot{y} + m\alpha\ddot{x} + ky + k\beta x = 0$ $\Rightarrow T = \begin{pmatrix} m & m\alpha \\ m\alpha & m \end{pmatrix}$ $V = \begin{pmatrix} k & k\beta \\ k\beta & k \end{pmatrix}$

$$T\ddot{\vec{x}} + V\vec{x} = 0$$

$$(-\omega^2 T + V)\vec{x} = 0 \Rightarrow$$

$$\det\begin{pmatrix} -\omega^2 m + k & -\omega^2 m\alpha + k\beta \\ -\omega^2 m\alpha + k\beta & -\omega^2 m + k \end{pmatrix} = (k - \omega^2 m)^2 - (k\beta - \omega^2 m\alpha)^2 = 0$$

$$= \omega^4 m^2(1-\alpha^2) - 2\omega^2 km(1-\beta\alpha) + k^2(1-\beta^2)$$

$$\Rightarrow \omega^2 = \frac{km(1-\beta\alpha) \pm \sqrt{k^2m^2(1-\beta\alpha)^2 - m^2(1-\alpha^2)k^2(1-\beta^2)}}{m^2(1-\alpha^2)} = \frac{k}{m}\frac{1-\beta\alpha \pm \sqrt{1-2\beta\alpha+\beta^2\alpha^2 - \cancel{1} + \alpha^2 + \beta^2 - \cancel{\alpha^2\beta^2}}}{1-\alpha^2}$$

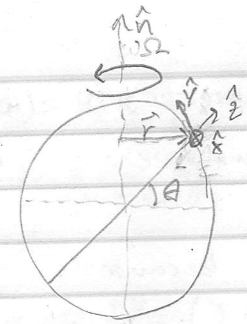
$$= \frac{k}{m}\frac{1-\beta\alpha \pm \alpha \mp \beta}{(1-\alpha)(1+\alpha)} = \frac{k}{m}\frac{(1\mp\beta)(1\pm\alpha)}{(1-\alpha)(1+\alpha)} = \frac{k}{m}\frac{1\mp\beta}{1\mp\alpha}$$

$$-\omega^2 T + V = k\begin{pmatrix} 1 - \frac{1\mp\beta}{1\mp\alpha} & \beta - \frac{1\mp\beta}{1\mp\alpha}\alpha \\ \beta - \frac{1\mp\beta}{1\mp\alpha}\alpha & 1 - \frac{1\mp\beta}{1\mp\alpha}\alpha \end{pmatrix} = \frac{k}{1\mp\alpha}\begin{pmatrix} \mp\alpha \pm\beta & \beta(1\mp\alpha) - \alpha(1\mp\beta) \\ \beta - \alpha & \pm(\beta-\alpha) \end{pmatrix} = k\frac{\beta-\alpha}{1\mp\alpha}\begin{pmatrix} \pm 1 & 1 \\ 1 & \pm 1 \end{pmatrix}$$

$\Rightarrow$ vectors $\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

2012

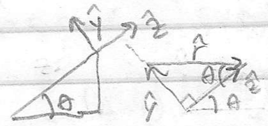
CM2



$\hat{x}$ east
$\hat{z}$ up

As the mass falls down the shaft, its tangential ($\hat{x}$) speed becomes faster than that of the shaft (the shaft is slower closer to the center of rotation) and it curves into the east wall of the shaft. This is the Coriolis force. Also, the centrifugal force causes the mass to curve into the south wall (assuming we start in northern hemisphere.).

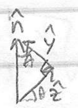
Let $\hat{x}$ point east, $\hat{y}$ point north, $\hat{z}$ point up.



$\hat{r}\cdot\hat{z} = \cos\theta, \quad \hat{r}\cdot\hat{y} = \sin\theta$

$\Rightarrow \hat{r} = \hat{z}\cos\theta - \hat{y}\sin\theta \qquad r = z\cos\theta$

$\dot{r} = \dot{z}\cos\theta - z\sin\theta\,\dot\theta$



$\hat{n}\cdot\hat{y} = \cos\theta, \quad \hat{n}\cdot\hat{z} = \sin\theta$

$\Rightarrow \hat{n} = \hat{z}\sin\theta + \hat{y}\cos\theta \qquad \vec{\Omega} = \Omega\hat{n}$

$$\vec{F} = -mg\left(\frac{z}{R}\right)^3\hat{z} - 2m\Omega\dot{r}\,\hat{n}\times\hat{r} - m\Omega^2 r\,\hat{n}\times(\hat{n}\times\hat{r})$$

$\hat{n}\times\hat{r} = \hat{x}, \quad \hat{n}\times\hat{x} = \hat{y}\sin\theta - \hat{z}\cos\theta = -\hat{r}, \quad \dot\theta = \frac{\dot{y}}{z}$

$$\vec{F} = -\frac{mg}{R^3}z^3\hat{z} - 2m\Omega\begin{pmatrix}\cos\theta\,\dot{z} \\ -\dot{y}\sin\theta\end{pmatrix}\hat{x} - m\Omega^2 z\cos\theta(\hat{y}\sin\theta - \hat{z}\cos\theta)$$

$\Rightarrow \ddot{x} = -2\Omega(\cos\theta\,\dot{z} - \sin\theta\,\dot{y})$

$\ddot{y} = -\Omega^2 z\cos\theta\sin\theta$

$\ddot{z} = -\frac{g}{R^3}z^3 + \Omega^2 z\cos^2\theta \qquad d = R - z$ dist. fallen

$\theta = 0: \ \ddot{x} = -2\Omega\dot{z}, \ \ddot{y} = 0, \ \ddot{z} = -\frac{g}{R^3}z^3 + \Omega^2 z$

~~no oscillation.~~ $\dot{z}$ is negative $\Rightarrow \ddot{x} > 0 \Rightarrow$ particle hits east wall.

$\theta \sim \frac{y}{z} \Rightarrow \ddot{x} = -2\Omega(\dot{z} - y/z\,\dot{y}), \ \ddot{y} = -\Omega^2 y \Rightarrow$ y coord oscillates with period Ω

$\theta = \pi/2: \ \ddot{x} = 2\Omega\dot{y}, \ \ddot{y} = 0, \ \ddot{z} = -\frac{g}{R^3}z^3$

$\pi/2 - \theta$ small, $y \approx z(\frac{\pi}{2} - \theta) \Rightarrow \cos\theta \simeq \frac{y}{z}$

$\ddot{x} = -2\Omega\left(\frac{y}{z}\dot{z} - \dot{y}\right), \quad \ddot{y} = -\Omega^2 y, \quad \ddot{z} = -\frac{g}{R^3}z^3 + \Omega^2 y^2/z$

$\Rightarrow$ y coord. oscillates with period Ω

$$z \simeq R \Rightarrow \ddot{z} = -g + \Omega^2 R\cos^2\theta \Rightarrow R - z = \frac{1}{2}(g - \Omega^2 R\cos^2\theta)t^2$$

$$\ddot{y} = -\Omega^2 R\cos\theta\sin\theta \qquad y = -\frac{1}{2}\Omega^2 R\cos\theta\sin\theta\, t^2$$

$$\ddot{x} = -2\Omega(\cos\theta\,\dot{z} - \sin\theta\,\dot{y})$$

$$= -2\Omega\left(\cos\theta(-g + \Omega^2 R\cos^2\theta) - \sin\theta(-\Omega^2 R\cos\theta\sin\theta)\right)t$$

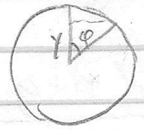
$$x = \frac{\Omega}{3}\left(g\cos\theta - \Omega^2 R\cos\theta\right)t^3 = \frac{\Omega}{3}(g - \Omega^2 R)\cos\theta\, t^3$$

$$x^2 + y^2 \equiv \alpha^2$$

$$\alpha^2 = \frac{\Omega^2}{9}(g - \Omega^2 R)^2\cos^2\theta\, t^6 + \frac{\Omega^4}{4}R^2\cos^2\theta\sin^2\theta\, t^4$$

$$t^2 = \frac{2d}{g - \Omega^2 R\cos^2\theta}$$

$$\alpha^2 = \frac{\Omega^2}{9}(g - \Omega^2 R)^2\cos^2\theta\left(\frac{2d}{g - \Omega^2 R\cos^2\theta}\right)^3 + \frac{\Omega^4}{4}R^2\cos^2\theta\sin^2\theta\left(\frac{2d}{g - \Omega^2 R\cos^2\theta}\right)^2$$



$$\sin\varphi = \frac{y}{\alpha} = -\frac{1}{2\alpha}\Omega^2 R\cos\theta\sin\theta\,\frac{2d}{g - \Omega^2 R\cos^2\theta}$$

$$\varphi = -\sin^{-1}\left(\frac{\Omega^2 R d\cos\theta\sin\theta}{\alpha(g - \Omega^2 R\cos^2\theta)}\right)$$

CM3 $\quad 2x\,dx + 2y\,dy - a\,dz = 0 \quad \rightarrow \quad x^2 + y^2 = \rho^2 \rightarrow 2\rho\,d\rho - a\,dz = 0$

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz \qquad L = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) - mgz$$

$$m\ddot{\rho} - m\rho\dot{\phi}^2 = 2\rho\lambda \qquad \xrightarrow{\frac{\lambda}{m}\to\lambda} \qquad \ddot{\rho} - \rho\dot{\phi}^2 = 2\lambda\rho$$

$$\frac{d}{dt}(m\rho^2\dot{\phi}) = 0 \qquad \Rightarrow \qquad \frac{d}{dt}(\rho^2\dot{\phi}) = 0$$

$$m\ddot{z} + mg = -a\lambda \qquad \ddot{z} = -g - a\lambda$$

$$az = \rho^2$$

take $\dot{\phi} = \sqrt{\frac{2g}{a}} \qquad l = \rho_0^2\sqrt{\frac{2g}{a}}, \quad \dot{\phi} = \frac{l}{\rho^2} \quad (l = \rho^2\dot{\phi})$

$$\ddot{\rho} - \frac{l^2}{\rho^3} = 2\lambda\rho, \quad \lambda = -\frac{\ddot{z} + g}{a}, \quad \dot{z} = \frac{1}{a}2\rho\dot{\rho}, \quad \ddot{z} = \frac{2}{a}(\dot{\rho}^2 + \rho\ddot{\rho})$$

$$\rho = \rho_0 + q \Rightarrow \ddot{q} - \frac{l^2}{\rho_0^3}(1 - 3q/\rho_0) = -\frac{2}{a}(\rho_0 + q)\left[g + \frac{2}{a}(\dot{q}^2 + (\rho_0 + q)\ddot{q})\right]$$

$$\ddot{q} + \frac{3l^2}{\rho_0^4}q - \frac{l^2}{\rho_0^3} = -\frac{2}{a}\rho_0 g - \frac{2}{a}gq - \frac{4}{a^2}\rho_0^2\ddot{q} \quad \text{(to 1st order)}$$

$$\Rightarrow \left(1 + \frac{4\rho_0^2}{a^2}\right)\ddot{q} + \left(\frac{3l^2}{\rho_0^4} + \frac{2g}{a}\right)q = 0 \Rightarrow \omega = \sqrt{\frac{\frac{3l^2}{\rho_0^4} + \frac{2g}{a}}{1 + 4\rho_0^2/a^2}}$$

2012 CM4 $L = \dot{q}_1 \dot{q}_2 - c q_1 q_2$

$\ddot{q}_2 + c q_2 = 0$ uncoupled oscillators

$\ddot{q}_1 + c q_1 = 0$

$$0 = \frac{dL}{d\lambda} = \frac{\partial L}{\partial q_i}\frac{dq_i}{d\lambda} + \frac{\partial L}{\partial \dot{q}_i}\frac{d\dot{q}_i}{d\lambda} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\frac{dq_i}{d\lambda} + \frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}\frac{dq_i}{d\lambda}$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\frac{\partial q_i}{\partial \lambda}\right), \quad \frac{\partial q_{1,2}}{\partial \lambda} = \pm q_{1,2}, \quad \frac{\partial L}{\partial \dot{q}_{1,2}} = \dot{q}_{2,1}$$

$\Rightarrow \dot{q}_2(q_1) + \dot{q}_1(-q_2)$ conserved

$\Rightarrow q_1 \dot{q}_2 - q_2 \dot{q}_1$ conserved – angular momentum

$$x\dot{y} - y\dot{x} = \frac{l}{m}$$

CM5 Equivalent to uniform gravitational field $a+g$

$E(t) = m(g+a)h(t)$, $J = \oint p\,dq = 2\int_0^h \sqrt{2m}\sqrt{E - m(g+a)y}\,dy$

$$J = 2\sqrt{2mE}\left(-\frac{2}{3}\right)\frac{E}{m(g+a)}\left(1 - \frac{m}{E}(g+a)y\right)^{3/2}\Big|_0^h = 2\sqrt{2mE}\,\frac{2}{3}h = \frac{4}{3}\sqrt{2(g+a)}\,m h^{3/2}$$

$$J = \text{const.} \Rightarrow h(t) = \text{const.}\cdot\frac{1}{(g+a)^{1/3}} = h_0\left(\frac{g+a(0)}{g+a(t)}\right)^{1/3}$$

SM1 $N = \int \frac{A\,d^2p}{h^2}\langle n_{p^2/2m}\rangle + (\varepsilon = 0)$, $\langle n_\varepsilon\rangle = \frac{1}{e^{\beta(\varepsilon-\mu)} - 1}$

$d^2p = 2\pi p\,dp$, $\varepsilon = p^2/2m \Rightarrow d\varepsilon = \frac{p}{m}dp$

$\Rightarrow d^2p = 2\pi m\,d\varepsilon$

$$N = \frac{2\pi A}{h^2} m \int_0^\infty \frac{d\varepsilon}{e^{\beta(\varepsilon-\mu)} - 1} + \frac{1}{e^{-\beta\mu} - 1}$$

$$= \frac{2\pi A m}{h^2 \beta}\underbrace{\int_0^\infty \frac{dx}{z^{-1}e^x - 1}}_{\Gamma(1) g_1(z) = g_1(z)} + \frac{1}{z^{-1} - 1}$$

$\Rightarrow N_0 = \frac{1}{z^{-1} - 1}$, $N_e = \frac{2\pi}{h^2} m A k T\, g_1(z)$

BEC: $z^{-1} = 1 + \frac{1}{N_0} \Rightarrow z = \left(1+\frac{1}{N_0}\right)^{-1} \simeq 1 - \frac{1}{N_0} \to 1$

$$\Rightarrow N_e = \frac{2\pi}{h^2} mAkT\, g_1(z) \Rightarrow T = \frac{N_e h^2}{2\pi m A k T\, g_1(z)} \to 0$$

since $g_1(1) = \zeta(1) = \infty$

SM2 $Q_N = \sum_{\{m_i\}} e^{\sum_{i=1}^N \beta g\mu_B m_i H} = \sum_{\{m_i\}} \prod_{i=1}^N e^{\beta g \mu_B m_i H} = \sum_{m_1=-j}^{j}\sum_{m_2=-j}^{j}\cdots\prod_{i=1}^N e^{\beta g\mu_B m_i H}$

$$= \prod_{i=1}^N \sum_{m_i=-j}^{j} e^{\beta g\mu_B m_i H} = \left(e^{-\beta g\mu_B j H}\underbrace{\sum_{m=0}^{2j} e^{\beta g \mu_B m H}}\right)^N$$

$$S_{2j} = 1+x+x^2+\dots+x^{2j} = xS_{2j}+1-x^{2j+1}$$

$$= \left(e^{-\beta g\mu_B jH}\frac{1-e^{\beta g\mu_B(2j+1)H}}{1-e^{\beta g\mu_B H}}\right)^N \quad \Rightarrow S_{2j}(1-x) = 1-x^{2j+1} \quad (x = e^{\beta g\mu_B H})$$

$$M_z = \langle\mu_z\rangle = \frac{1}{\beta}\frac{\partial}{\partial H}\log Q_N = \frac{1}{\beta}\frac{\partial}{\partial H} N\left(-\beta g\mu_B jH + \log\left(1-e^{\beta g\mu_B(2j+1)H}\right) - \log\left(1-e^{\beta g\mu_B H}\right)\right)$$

$$= \frac{N}{\beta}\left(-\beta g\mu_B j + \frac{-\beta g\mu_B(2j+1)e^{\beta g\mu_B(2j+1)H}}{1-e^{\beta g\mu_B(2j+1)H}} - \frac{-\beta g\mu_B e^{\beta g\mu_B H}}{1-e^{\beta g\mu_B H}}\right)$$

$$= Ng\mu_B\left(-j - \frac{(2j+1)e^{(2+\frac{1}{j})x}}{1-e^{(2+1/j)x}} + \frac{e^{x/j}}{1-e^{x/j}}\right) \qquad x \equiv g\mu_B H j\beta$$

$$= -Ng\mu_B j\left(1 + \frac{2(1+\frac{1}{2j})e^{2(1+\frac{1}{2j})x}}{1-e^{2(1+1/2j)x}} - \frac{2}{2j}\frac{e^{2x/2j}}{1-e^{2x/2j}}\right)$$

$$= -Ng\mu_B j\left[\left(1+\frac{1}{2j}\right)\left\{\frac{\overbrace{-\sinh((1+\frac{1}{2j})x)+e^{(1+\frac{1}{2j})x}}^{\cosh((1+\frac{1}{2j})x)}}{-\sinh((1+\frac{1}{2j})x)}\right\} - \frac{1}{2j}\left\{\frac{\overbrace{-\sinh(\frac{x}{2j})+e^{x/2j}}^{\cosh(x/2j)}}{-\sinh(x/2j)}\right\}\right]$$

$$= Ng\mu_B j B_j(x)$$

$x \ll 1$: $\chi = \frac{\partial M_z}{\partial H} = Ng\mu_B j \cdot g\mu_B j\beta\, \frac{dB_j(x)}{dx} = Ng^2\mu_B^2 j\beta\, \frac{dB_j}{dx}$

$$\chi = Ng^2\mu_B^2 j\beta\left[\cancel{\frac{1}{x}} + \frac{1}{3}\left(1+\frac{1}{2j}\right)^2 x - \cancel{\frac{1}{x}} - \frac{1}{3}\left(\frac{1}{2j}\right)^2 x\right] = \frac{Ng^3\mu_B^3 j^2(j+1)}{3k_BT}$$

2012 SM3 $Q_N = V^N + V^{N-2} \sum_{i<k} \int_V d^3 r_1 \int_{|\vec{r}_i - \vec{r}_k| < r_0} d^3 r_k (-1)$

$$= V^N - V^{N-2} \sum_{i<k} \int d^3 r_1 \frac{4}{3}\pi r_0^3 = V^N - V^{N-2} \underbrace{\sum_{i<k}}_{\frac{N(N-1)}{2}} V \cdot \frac{4\pi}{3} r_0^3$$

$$= V^N - V^{N-1} \frac{2\pi (N-1)N}{3} r_0^3 = V^N \left(1 - \frac{2\pi (N-1)N}{3V} r_0^3 \right)$$

$$A = -k \log Z = -kT\left(-\log N! - 3N \log \lambda + N \log V + \log\left(1 - \frac{2\pi(N-1)N}{3V} r_0^3\right)\right)$$

$$P = -\frac{\partial A}{\partial V} = kT\left(\frac{N}{V} + \frac{\frac{2\pi (N-1) N}{3V^2} r_0^3}{\underbrace{1 - \frac{2\pi N(N-1)}{3V} r_0^3}} \right), \quad N >> 1$$

$$= kT \left(\frac{N}{V} + \frac{2\pi r_0^3}{3} \frac{N^2}{V^2} \right) \quad \text{small because } N r_0^3 << V/N$$

$$= \frac{NkT}{V} \left(1 + \frac{2\pi r_0^3}{3} \frac{N}{V} \right) \Rightarrow PV \left(1 - \frac{2\pi r_0^3}{3} \frac{N}{V} \right) = NkT$$

$$\Rightarrow P\left(V - N \frac{2\pi}{3} r_0^3\right) = NkT$$

SM4 $Q_N = \sum_{\sigma_1 = \pm 1, \dots, \sigma_N = \pm 1} e^{\beta I \sum_{i=1}^N \sigma_i \sigma_{i+1} + \beta\mu B \sum_{i=1}^N \sigma_i}$ → write as $\frac{1}{2} \sum_{i=1}^N (\sigma_i + \sigma_{i+1})$

$$= \sum_{\sigma_1, \dots, \sigma_N = \pm 1} \prod_{i=1}^N e^{\beta I \sigma_i \sigma_{i+1} + \frac{1}{2}\beta \mu B (\sigma_i + \sigma_{i+1})}$$

$$T = \begin{array}{c|cc} & + & - \\ \hline + & e^{\beta I + \beta\mu B} & e^{-\beta I} \\ - & e^{-\beta I} & e^{\beta I - \beta \mu B} \end{array}$$

$$= \sum_{\sigma_1, \dots, \sigma_N = \pm 1} \prod_{i=1}^N T_{\sigma_i \sigma_{i+1}} = \sum_{\sigma_1, \dots, \sigma_N} T_{\sigma_1 \sigma_2} T_{\sigma_2 \sigma_3} \cdots T_{\sigma_{N-1} \sigma_N} T_{\sigma_N \sigma_1}$$

$= \text{tr}\, T^N = \lambda_1^N + \lambda_2^N$, $\lambda_{1,2}$ eigenvalues of T

$$\det \begin{pmatrix} e^{\beta I + \beta \mu B} - \lambda & e^{-\beta I} \\ e^{-\beta I} & e^{\beta I - \beta \mu B} - \lambda \end{pmatrix} = \left(e^{\beta I + \beta \mu B} - \lambda\right)\left(e^{\beta I - \beta \mu B} - \lambda\right) - e^{-2\beta I} = e^{2\beta I} - \lambda e^{\beta I}\left(e^{\beta \mu B} + e^{-\beta \mu B}\right) + \lambda^2 - e^{-2\beta I}$$

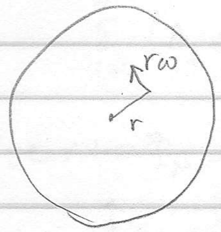
$$0 = \lambda^2 - 2\lambda e^{\beta I} \cosh \beta \mu B + e^{2\beta I} - e^{-2\beta I}$$

$$\lambda_{1,2} = e^{\beta I} \cosh \beta \mu B \pm \sqrt{e^{2\beta I} \cosh^2 \beta \mu B - e^{2\beta I} + e^{-2\beta I}}$$

$$\lambda_{1,2} = e^{\beta I}\cosh\beta\mu B \pm \sqrt{e^{-2\beta I} + e^{2\beta I}\sinh^2\beta\mu B}\ , \quad \beta = \frac{1}{kT}$$

$$A = -kT\log Q_N = -kT\log\left(\lambda_1^N + \lambda_2^N\right)$$

SM5



$$Q_1 = \sum_\varepsilon e^{-\beta\varepsilon}, \quad \varepsilon = \frac{p^2}{2m} + \frac{mr^2}{2}\omega^2 \qquad (p = p_{free})$$

$$\sum_\varepsilon \to \int d\Sigma'(\varepsilon)$$

$$d\Sigma = \frac{d^3x\, d^3p}{h^3}$$

$$Q_1 = \frac{1}{h^3}\int_0^\infty 4\pi p^2 dp \int_0^R 2\pi r L\, dr\; e^{-\beta\left(\frac{p^2}{2m} + \frac{m\omega^2}{2}r^2\right)}, \quad V = \pi R^2 L$$

$$= \underbrace{\frac{V}{h^3}\int_0^\infty 4\pi p^2 dp\, e^{-\beta p^2/2m}}_{Q_1^{free}}\ \underbrace{\frac{2\pi L}{V}\int_0^R r\,dr\, e^{-\beta\frac{m\omega^2}{2}r^2}}_{Q'}$$

$$Q' = \frac{1}{R^2}\int_0^R d(r^2)\, e^{-\beta\frac{m\omega^2}{2}r^2} = \frac{1}{R^2}\frac{-2}{\beta m\omega^2}e^{-\beta\frac{m\omega^2}{2}r^2}\Big|_0^R$$

$$= \frac{2}{\beta m\omega^2 R^2}\left(1 - e^{-\beta\frac{m\omega^2}{2}R^2}\right)$$

$$Q_N = \frac{Q_1^N}{N!} = \frac{(Q_1^{free})^N}{N!}(Q')^N$$

$$\Delta A = kT\log Q_N - kT\log\left(\frac{(Q_1^{free})^N}{N!}\right) = kT\log(Q')^N$$

$$= NkT\left[\log\left(\frac{2}{\beta m\omega^2 R^2}\right) + \log\left(1 - e^{-\beta\frac{m\omega^2}{2}R^2}\right)\right]$$

2012 EM 1-1 $\frac{\partial \rho}{\partial t} + \nabla\cdot\vec{j} = e\, \underbrace{\nabla_R \delta(\vec{r}-\vec{R}(t))}_{= -\nabla_r \delta(\vec{r}-R(t))} \cdot \frac{d\vec{R}}{dt} + e \frac{d\vec{R}}{dt}\cdot\nabla_r \delta(\vec{r}-\vec{R}(t)) = 0$

$\rho = -\vec{d}\cdot\nabla\delta(\vec{r})$

$Q = -\vec{d}\cdot\int d^3r\, \nabla\delta(\vec{r}) = 0$

$\vec{p} = -\int d^3r\, \vec{r}\, d_i\, \partial_i \delta(\vec{r})$

$= d_i \int d^3r\, \delta(\vec{r})\, \partial_i \vec{r} = \int d^3\vec{r}\, \delta(\vec{r})\, \vec{d} = \vec{d}$

$\Phi = 0,\ \hat{A} = \hat{a}(\hat{a}\cdot\hat{r})$

$\vec{E} = -\nabla\Phi - \frac{\partial\hat{A}}{\partial t} = -\frac{\partial}{\partial t}\hat{a}(\hat{a}\cdot\hat{r}) = 0$

$\hat{B} = \nabla\times\hat{A} = \hat{k}\epsilon_{ijk}\, \partial_i\, a_j\, a_l\, r_l$

$= \hat{k}\, \epsilon_{ijk}\, a_j a_l \delta_{il} = \hat{k}\, \epsilon_{ijk}\, a_i a_j = 0$

EM1-2 $\frac{1}{|\vec{r}-\vec{r}'|} = \left[(\vec{r}-\vec{r}')^2\right]^{-1/2} \simeq \left[r^2 - 2\vec{r}\cdot\vec{r}'\right]^{-1/2} \simeq r^{-1}\left(1-\frac{2}{r^2}\vec{r}\cdot\vec{r}'\right)^{-1/2}$

$\simeq \frac{1}{r}\left(1+\frac{\vec{r}\cdot\vec{r}'}{r^3}\right)$

$\phi(\vec{r}) \simeq \int d^3r'\, \rho(\vec{r}')\left(\frac{1}{r}+\frac{\vec{r}\cdot\vec{r}'}{r^3}\right)$

$\simeq \frac{1}{r}\underbrace{\int d^3r' \rho(\vec{r}')}_{Q=0} + \frac{\vec{r}}{r^3}\cdot\underbrace{\int d^3\vec{r}'\, \rho(\vec{r}')\, \vec{r}'}_{\vec{d}} = \frac{\vec{r}\cdot\vec{d}}{r^3}$

$E = \vec{d}\cdot\frac{e_1\vec{r}}{r^3}$, but $\vec{E} = \frac{-e_1\vec{r}}{r^3}$ at the origin due to e_1

$\Rightarrow E = -\vec{d}\cdot\vec{E}$

Dipole: $\vec{E}_2 = -\nabla\frac{\vec{r}_{12}\cdot\vec{d}_2}{r_{12}^3} = -d_{2i}\hat{j}\partial_j \frac{r_{12i}}{(\Sigma r_{12k}^2)^{3/2}}$

$= -d_{2i}\hat{j}\, \frac{r_{12}^3\delta_{ij} - 3 r_{12i} r_{12j} r}{r_{12}^6} = -d_{2i}\, \frac{r_{12}^2\hat{i} - 3 r_{12i}\vec{r}_{12}}{r_{12}^5} = -\frac{\vec{d}_2}{r_{12}^3} + 3\frac{(\vec{d}_2\cdot\vec{r}_{12})\vec{r}_{12}}{r_{12}^5}$

$$E = -\vec{d}_1 \cdot \vec{E}_2 = \frac{\vec{d}_1 \cdot \vec{d}_2}{r^3} - 3\,\frac{(\vec{d}_1 \cdot \vec{r})(\vec{d}_2 \cdot \vec{r})}{r^5}$$

EM1-3 $\vec{B}_0 = B_0\,\hat{z}$

No current outside sphere $\Rightarrow$ write $\vec{B} = -\nabla \Phi_M$.

$$\Phi_M = \sum_{l=0}^{\infty} \left(A_l r^l + C_l r^{-l-1}\right) P_l(\cos\theta) = \sum_{l=0}^{\infty} C_l r^{-l-1} P_l(\cos\theta) - B_0 r\cos\theta$$

large r: $\Phi_M = -B_0 r\cos\theta \Rightarrow A_1 = -B_0$, other $A = 0$

$$B_r\Big|_{r=a} = -\frac{\partial\Phi}{\partial r}\Big|_{r=a} = B_0\cos\theta + \sum_{l=0}^{\infty} C_l (l+1) a^{-l-2} P_l(\cos\theta) = 0$$

$$\Rightarrow B_0 = -C_1 \cdot 2a^{-3}, \quad \text{other } C = 0 \quad \Rightarrow C_1 = \frac{-B_0}{2} a^3$$

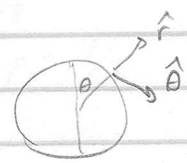
$$\Phi_M = -B_0 r\cos\theta - \underbrace{\frac{B_0}{2}\frac{a^3}{r^2}\cos\theta}_{\frac{\vec{r}\cdot\vec{\mu}}{r^3}}$$

$$\Rightarrow \hat{r}\cdot\vec{\mu} = \frac{-B_0}{2} a^3 \cos\theta \Rightarrow \vec{\mu} = \frac{-B_0 a^3}{2}\hat{z}$$

$$\Rightarrow \vec{\mu} = -\frac{a^3}{2}\vec{B}_0.$$

surface: $\hat{r} \times \frac{\vec{B}}{\mu_0}\Big|_{r=a} = \vec{k}$

$$\vec{B}\Big|_{r=a} = -\nabla\Phi\Big|_{r=a} = \left(-\frac{\partial\Phi_M}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial\Phi_M}{\partial\theta}\hat{\theta}\right)\Big|_{r=a}$$



$$= -\frac{1}{a}\left(B_0 a\sin\theta + \frac{B_0}{2} a\sin\theta\right)\hat{\theta} = -\frac{3}{2} B_0 \sin\theta\,\hat{\theta}$$

$$\Rightarrow \vec{k} = \frac{1}{4\pi}\hat{r}\times\vec{B} = \frac{-1}{4\pi}\frac{3}{2} B_0 \sin\theta\,\hat{\varphi}$$

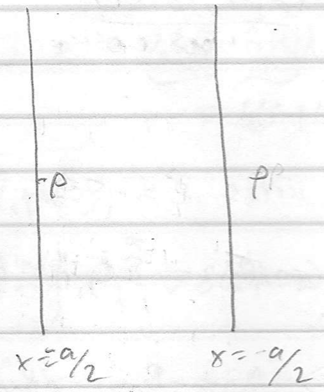
$$\vec{\mu} = \frac{1}{2}\int_V d^3x'\,\vec{x}'\times\vec{J} = \frac{1}{2}a\int_S d^2x'\,\vec{x}'\times\vec{k} = -\frac{3}{4}B_0\int_{-1}^{1} a^2 \frac{1}{2} d\mu\,(-\hat{\theta})\,a\sin\theta$$

$\mu = \cos\theta$

but $\hat{\theta} = \hat{\rho}\cos\theta - \hat{z}\sin\theta \Rightarrow \vec{\mu} = \frac{3}{8}B_0 a^3 \int_{-1}^{1} d\mu\,(\cancel{\hat{\rho}\mu} - \hat{z}\sqrt{1-\mu^2})\sqrt{1-\mu^2}$

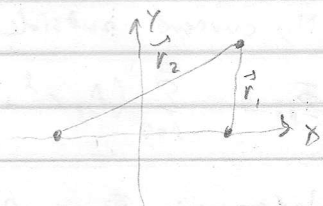
$$\vec{\mu} = -\frac{3}{8}B_0 a^3 \underbrace{\int_{-1}^{1}(1-\mu^2)\,d\mu}_{2-\frac{2}{3}=\frac{4}{3}}\,\hat{z} = -\frac{1}{2}B_0 a^3\hat{z} = -\frac{a^3}{2}\vec{B}_0$$

2012 EM1-4



one wire: $2\pi r E = 4\pi\rho$

$$\Rightarrow \vec{E} = \frac{2\rho}{r}\hat{r} = \frac{2\rho}{r^2}\vec{r}$$



$\vec{r}_1 = (x - a/2)\hat{x} + y\hat{y}$, $\vec{r}_2 = (x + \frac{a}{2})\hat{x} + y\hat{y}$

$$\vec{E}_{1,2} = \frac{\pm 2\rho}{(x \mp a/2)^2 + y^2}\left[(x \mp a/2)\hat{x} + y\hat{y}\right]$$

$$\vec{E} = 2\rho\left[\frac{(x-a/2)\hat{x} + y\hat{y}}{(x-a/2)^2 + y^2} - \frac{(x+a/2)\hat{x} + y\hat{y}}{(x+a/2)^2 + y^2}\right]$$

$$= 2\rho\left\{\left[\frac{x - a/2}{(x-a/2)^2 + y^2} - \frac{x + a/2}{(x+a/2)^2 + y^2}\right]\hat{x} + y\left[\frac{1}{(x-a/2)^2 + y^2} - \frac{1}{(x+a/2)^2 + y^2}\right]\hat{y}\right\}$$

one wire:

$$\nabla\Phi = \frac{\partial\Phi}{\partial r}\hat{r} = \frac{-2\rho}{r}\hat{r} \Rightarrow \Phi = -2\rho\log r + C$$

both:

$$\Phi = -2\rho\left[\log\sqrt{(x-a/2)^2 + y^2} - \log\sqrt{(x+a/2)^2 + y^2}\right]$$

$$= \rho\log\frac{(x+a/2)^2 + y^2}{(x-a/2)^2 + y^2}$$

far away: $(x \pm a/2)^2 + y^2 \simeq x^2 + y^2 \pm ax$

$$\Phi \simeq \rho\log\frac{x^2 + y^2 + ax}{x^2 + y^2 - ax} = \rho\log\left(\frac{1 + ax/x^2+y^2}{1 - ax/x^2+y^2}\right) \simeq \rho\log\left(1 + \frac{2ax}{x^2+y^2}\right)$$

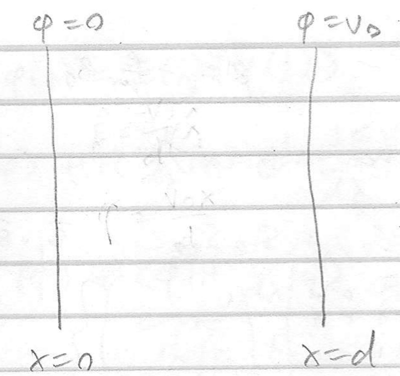
$$\simeq \rho \cdot \frac{2ax}{x^2+y^2} = \frac{2\rho x}{x^2+y^2}$$

$$\vec{E} \simeq 2\rho\frac{1}{x^2+y^2}\left\{x\left[\frac{1 - \frac{a}{2x}}{1 - \frac{ax}{x^2+y^2}} - \frac{1 + \frac{a}{2x}}{1 + \frac{ax}{x^2+y^2}}\right]\hat{x} + y\left[\frac{1}{1 - \frac{ax}{x^2+y^2}} - \frac{1}{1 + \frac{ax}{x^2+y^2}}\right]\hat{y}\right\}$$

$$\simeq \frac{2\rho}{x^2+y^2}\left\{x\left[-\frac{a}{x} + \frac{2ax}{x^2+y^2}\right]\hat{x} + y\frac{2ax}{x^2+y^2}\hat{y}\right\} = \frac{-2\rho}{r^2}\hat{x} + \frac{4\rho x}{r^3}\hat{r}$$

where $\vec{r} = x\hat{x} + y\hat{y}$ (cylindrical r)

EMI-5



One electron:

$T = \frac{1}{2}mv^2, \quad U = -e\varphi$

$E = \frac{1}{2}mv^2 - e\varphi = 0$

$\Rightarrow v^2 = \frac{2e}{m}\varphi$

$j = \rho v = \rho\sqrt{\frac{2e}{m}\varphi}$

$0 = \frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x}$ (conservation of charge), but in steady state

$\frac{\partial \rho}{\partial t} = 0 \Rightarrow \frac{\partial j}{\partial x} = 0 \Rightarrow j$ is const. in x

$\nabla^2 \varphi = \frac{\partial^2}{\partial x^2}\varphi = -4\pi\rho$, but $\rho = \frac{j}{\sqrt{\frac{2e}{m}\varphi}} = j\sqrt{\frac{m}{2e}}\,\varphi^{-1/2}$

$\Rightarrow \frac{\partial^2}{\partial x^2}\varphi = -4\pi j\sqrt{\frac{m}{2e}}\,\varphi^{-1/2}$

$\varphi = ax^p \Rightarrow \frac{\partial^2}{\partial x^2}\varphi = p(p-1)\,ax^{p-2}$

$\varphi^{-1/2} = a^{-1/2}x^{-p/2}$

$p(p-1)\,ax^{p-2} = -4\pi j\sqrt{\frac{m}{2e}}\,a^{-1/2}x^{-p/2} \quad \Rightarrow p-2 = -p/2$

$\Rightarrow \frac{3}{2}p = 2 \Rightarrow p = 4/3$

$\frac{4}{9}a^{3/2} = -4\pi j\sqrt{\frac{m}{2e}}$

$\Rightarrow a = \left((9\pi j)^2\frac{m}{2e}\right)^{1/3}$

$\varphi = \left(81\pi^2 j^2\frac{m}{2e}\right)^{1/3} x^{4/3}$

$V_0 = \left(81\pi^2 j^2\frac{m}{2e}\right)^{1/3} d^{4/3} \Rightarrow \frac{V_0^3}{81\pi^2}\frac{2e}{m}\frac{1}{d^4} = j^{2/3}$

$\Rightarrow j = \left(\frac{V_0^3}{81\pi^2}\frac{2e}{m}\frac{1}{d^4}\right)^{3/2} = \left(\frac{2e}{m}V_0^3\right)^{3/2}\left(\frac{1}{9\pi d^2}\right)^3$

2012 QMII-1 $H = \alpha \vec{S}_1 \cdot \vec{S}_2 + \frac{\beta}{B_0} \vec{S}_1 \cdot \vec{B}_0 \qquad \beta = -\frac{e}{m} B_0$

$\alpha < 0$ (lower energy when aligned)

$\vec{S}_1 \cdot \vec{S}_2 = S_{1x} S_{2x} + S_{1y} S_{2y} + S_{1z} S_{2z} \qquad \vec{S}_1 \cdot \vec{B}_0 = S_{1z} B_0$

$$= \frac{\hbar^2}{4}\left\{\begin{pmatrix} 0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix} + \begin{pmatrix} 0&0&0&-1\\0&0&1&0\\0&1&0&0\\-1&0&0&0\end{pmatrix} + \begin{pmatrix} 1&&&\\&-1&&\\&&-1&\\&&&1\end{pmatrix}\right\} \qquad = \frac{\hbar}{2}\begin{pmatrix} 1&&&\\&1&&\\&&-1&\\&&&-1\end{pmatrix}$$

$$H = \frac{\alpha\hbar^2}{4}\begin{pmatrix} 1&0&0&0\\0&-1&2&0\\0&2&-1&0\\0&0&0&1\end{pmatrix} + \frac{\beta\hbar}{2}\begin{pmatrix} 1&&&\\&1&&\\&&-1&\\&&&-1\end{pmatrix}$$

$$= \frac{\hbar}{4}\begin{pmatrix} \alpha\hbar+2\beta & 0 & 0 & 0\\ 0 & -\alpha\hbar+2\beta & 2\alpha\hbar & 0\\ 0 & 2\alpha\hbar & -\alpha\hbar-2\beta & 0\\ 0 & 0 & 0 & \alpha\hbar-2\beta\end{pmatrix} \qquad E = \frac{\hbar}{4}\lambda$$

$$(\alpha\hbar+2\beta-\lambda)(-\alpha\hbar+2\beta-\lambda)(-\alpha\hbar-2\beta-\lambda)(\alpha\hbar-2\beta-\lambda)$$
$$- (\alpha\hbar+2\beta-\lambda)(\alpha\hbar-2\beta-\lambda)\,4\alpha^2\hbar^2 = 0$$

$$\left((\alpha\hbar-\lambda)^2-(2\beta)^2\right)\left[(\alpha\hbar+\lambda)^2-(2\beta)^2 - (2\alpha\hbar)^2\right] = 0$$

$$\Rightarrow \alpha\hbar-\lambda = \pm 2\beta, \quad \alpha\hbar-\lambda = \pm\sqrt{(2\alpha\hbar)^2+(2\beta)^2}$$

$$\lambda = a\hbar \pm 2\beta, \quad a\hbar \pm \sqrt{(2\alpha\hbar)^2+(2\beta)^2}$$

$$E = \frac{\hbar^2 a}{4} \pm \frac{\hbar^2 e B_0}{2m} \quad \text{or} \quad E = \frac{\hbar^2 a}{4} \pm \hbar\sqrt{(\alpha\hbar)^2 + \left(\frac{eB_0}{m}\right)^2}$$

$$\rho = |+\rangle\langle +| \otimes \tfrac{1}{2}\mathbb{1} = \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} \otimes \begin{pmatrix}1/2 & \\ & 1/2\end{pmatrix} = \begin{pmatrix}1&0\\0&0\end{pmatrix} \otimes \begin{pmatrix}1/2 & 0\\ 0 & 1/2\end{pmatrix}$$

$$= \begin{pmatrix} 1/2 & 0 & & \\ 0 & 1/2 & & 0 \\ & & & \\ 0 & & & 0\end{pmatrix}$$

$$[H] = \mathrm{tr}\left(\begin{pmatrix} 1/2&0&0&0\\0&1/2&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} \frac{\hbar}{4}\begin{pmatrix} \alpha\hbar+2\beta & 0 & 0 & 0\\ 0 & -\alpha\hbar+2\beta & 2\alpha\hbar & 0\\ 0 & 2\alpha\hbar & -\alpha\hbar-2\beta & 0\\ 0 & 0 & 0 & \alpha\hbar-2\beta\end{pmatrix}\right)$$

$$= \mathrm{tr}\,\frac{\hbar}{8}\begin{pmatrix} \alpha\hbar+2\beta & 0 & 0 & 0\\ 0 & -\alpha\hbar+2\beta & 2\alpha\hbar & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix} = \frac{\hbar}{8}4\beta$$

$$= -\frac{\hbar}{2}\frac{eB_0}{m}$$

QM1-2 $H(\lambda)|\varphi(\lambda)\rangle = E(\lambda)|\varphi(\lambda)\rangle$

$$\frac{dH}{d\lambda}|\varphi\rangle + H\frac{d}{d\lambda}|\varphi\rangle = \frac{dE}{d\lambda}|\varphi\rangle + E\frac{d}{d\lambda}|\varphi\rangle$$

$$\langle\varphi|\frac{dH}{d\lambda}|\varphi\rangle + \cancel{\underbrace{\langle\varphi|H}_{\langle\varphi|E}\frac{d}{d\lambda}|\varphi\rangle} = \underbrace{\langle\varphi|\frac{dE}{d\lambda}|\varphi\rangle}_{dE/d\lambda} + \cancel{\langle\varphi|E\frac{d}{d\lambda}|\varphi\rangle}$$

$\Rightarrow$

$$\langle\varphi|\frac{dH}{d\lambda}|\varphi\rangle = \frac{dE}{d\lambda}$$

$$H = \frac{l(l+1)\hbar^2}{2mr^2} + \ldots \qquad E = \frac{-e^4 m}{2\hbar^2(l+1)^2}$$

$$\frac{dH}{dl} = (2l+1)\frac{\hbar^2}{2mr^2} \qquad \frac{dE}{dl} = \frac{e^4 m}{\hbar^2(l+1)^3}$$

$$\langle\varphi(l)|r^{-2}|\varphi(l)\rangle = \frac{2m}{(2l+1)\hbar^2}\langle\varphi(l)|\frac{dH(l)}{dl}|\varphi(l)\rangle = \frac{2m}{(2l+1)\hbar^2}\frac{dE}{dl}$$

$$= \frac{2m^2e^4}{\hbar^4(2l+1)(l+1)^3}$$

QM1-3 $H = \begin{pmatrix} 0 & \epsilon & \epsilon \\ \epsilon & 0 & \epsilon \\ \epsilon & \epsilon & 0 \end{pmatrix}$ $\quad \det(H-E) = -E^3 + E\epsilon^2 + \epsilon^3 + E\epsilon^2 + \epsilon^3 + E\epsilon^2$

$$= -E^3 + 3E\epsilon^2 + 2\epsilon^3 = 0$$

$R|S_i\rangle = |S_{i+1}\rangle$ $\quad = -(E-2\epsilon)(E^2+2\epsilon E+\epsilon^2) = 0$

i.e. $R\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix}$ $\quad = -(E-2\epsilon)(E+\epsilon)^2 = 0 \Rightarrow E = 2\epsilon,\ E = -\epsilon$

$R\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix},\ R\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}$ $\qquad$ mult. 2

$\Rightarrow R = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \to$ eigenstates $\underbrace{\frac{1}{\sqrt3}\begin{pmatrix}1\\1\\1\end{pmatrix}}_{|\psi_1\rangle,\ \text{energy } 2\epsilon},\ \underbrace{\frac{1}{\sqrt3}\begin{pmatrix}1\\e^{2\pi/3 i}\\e^{4\pi/3 i}\end{pmatrix},\ \frac{1}{\sqrt3}\begin{pmatrix}1\\e^{4\pi/3 i}\\e^{2\pi/3 i}\end{pmatrix}}_{|\psi_2\rangle,\ |\psi_3\rangle:\ \text{energy } -\epsilon}$

$$|S_1\rangle = \frac{1}{\sqrt3}(|\psi_1\rangle + |\psi_2\rangle + |\psi_3\rangle)$$

$$|\langle S_1|e^{-iH/\hbar t}|S_1\rangle|^2 = \left|\langle S_1|\frac{1}{\sqrt3}\left(e^{-i2\epsilon/\hbar t}|\psi_1\rangle + e^{i\epsilon/\hbar t}|\psi_2\rangle + e^{i\epsilon/\hbar t}|\psi_3\rangle\right)\right|^2$$

$$= \left|\frac{1}{3}\left(e^{-i\frac{2\epsilon t}{\hbar}} + 2e^{i\epsilon/\hbar t}\right)\right|^2 = \frac{1}{9}\left[\left(2+\cos\left(\frac{3\epsilon t}{\hbar}\right)\right)^2 + \sin^2\left(\frac{3\epsilon t}{\hbar}\right)\right] = \frac{5+4\cos\left(\frac{3\epsilon t}{\hbar}\right)}{9}$$

2012 QM1-4 $\mathrm{tr}\,[A,B] = \mathrm{tr}\,(AB-BA) = \mathrm{tr}(AB) - \mathrm{tr}(BA)$

$$= A_{ij}B_{ji} - B_{ij}A_{ji} = A_{ij}B_{ji} - B_{ji}A_{ij} = 0$$

$a|n\rangle = \sqrt{n}\,|n-1\rangle\,, \quad a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$

$$a = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{4} \\ 0 & 0 & 0 & 0 & 0 \\ & & & & & \ddots \end{pmatrix} \qquad a^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{4} & 0 \\ & & & & & \ddots \end{pmatrix}$$

$$aa^\dagger = \begin{pmatrix} 1 & & & & \\ & 2 & & & \\ & & 3 & & \\ & & & 4 & \\ & & & & \ddots \end{pmatrix} \qquad a^\dagger a = \begin{pmatrix} 0 & & & & \\ & 1 & & & \\ & & 2 & & \\ & & & 3 & \\ & & & & \ddots \end{pmatrix}$$

Because the matrices are infinite dimensional, their traces diverge. So even though they are the same formal sum $\sum_{i=1}^{\infty} i$, the subtraction between them (as above) is ill defined.

QM1-5 $\frac{\partial\rho}{\partial t} = -\frac{1}{i\hbar}[\rho, H]$ pure state: $\rho^2 = \rho$

$$\frac{\partial}{\partial t}(\rho^2 - \rho) = \rho\frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial t}\rho - \frac{\partial\rho}{\partial t} = -\frac{1}{i\hbar}\Big(\underbrace{\rho[\rho,H] + [\rho,H]\rho}_{[\rho^2,H]} - [\rho,H]\Big)$$

$$= -\frac{1}{i\hbar}\left([\rho^2 - \rho, H]\right)$$

$$= 0 \text{ for a pure state}$$

$\Rightarrow$ pure state cannot evolve into mixed state

$\Rightarrow$ mixed state cannot evolve into pure state (by time symmetry)

$$\rho = \frac{1}{2}\left(|+-\rangle - |-+\rangle\right)\left(\langle +-| - \langle -+|\right) = \frac{1}{2}\left(|+-\rangle\langle +-| - |-+\rangle\langle +-| - |+-\rangle\langle -+| + |-+\rangle\langle -+|\right)$$

$$\rho_R = {}_2\langle +|\rho|+\rangle_2 + {}_2\langle -|\rho|-\rangle_2 = \frac{1}{2}\left(|+\rangle_1\langle +| + |-\rangle_1\langle -|\right)$$

mixed, unpolarized

EMII-1 $P = \frac{2}{3}\frac{e^2}{c^3}|\dot{v}|^2$; Neglect energy loss ($P\Delta t \ll E$):

so $m\dot{\vec{v}} = -\frac{\partial V}{\partial r}\hat{r} \Rightarrow |\dot{v}|^2 = \frac{1}{m^2}\left(\frac{\partial V}{\partial r}\right)^2$ and $P = \frac{2e^2}{3c^3m^2}\left(\frac{\partial V(r)}{\partial r}\right)^2$

$$\int_{-\infty}^{\infty} P\,dt = \frac{2e^2}{3m^2c^3}\,2\int_{r_0}^{\infty}\left(\frac{\partial V}{\partial r}\right)^2\frac{dr}{v(r)}, \quad v(r) = \sqrt{\frac{2}{m}(E-V(r))}$$

$\to 0$ since $V(r_0)=E$, $\frac{\partial V}{\partial r}\big|_\infty = 0$

$$= \frac{2}{3}\frac{e^2}{m^2c^3}\,2\sqrt{\frac{m}{2}}\int_{r_0}^{\infty}\frac{\partial V}{\partial r}\underbrace{\frac{V'\,dr}{\sqrt{E-V}}}_{d(-2\sqrt{E-V})} = \frac{2}{3}\frac{e^2}{m^2c^3}\,2\sqrt{\frac{m}{2}}(-2)\left[\cancel{\sqrt{E-V}\frac{\partial V}{\partial r}\Big|_{r_0}^{\infty}} - \int_{r_0}^{\infty}\sqrt{E-V}\frac{\partial^2 V}{\partial r^2}dr\right]$$

$$= \frac{2^{5/2}e^2}{3m^{3/2}c^3}\int_{r_0}^{\infty}\underbrace{\sqrt{E-V(r)}}_{\le\sqrt{E-V_{min}}}\frac{\partial^2 V(r)}{\partial r^2}dr$$

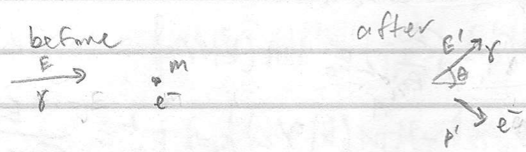
V_{min} = smallest value of V for $r \geq r_0$

$$\left|\int_{-\infty}^{\infty} P\,dt\right| \le (\text{const})\left|\int_{r_0}^{\infty}\frac{\partial^2 V(r)}{\partial r^2}dr\right| = (\text{const})\left|\underbrace{\frac{\partial V(r)}{\partial r}\Big|_{r_0}^{\infty}}_{\frac{\partial V}{\partial r}\big|_\infty - \frac{\partial V}{\partial r}\big|_{r_0} = -\frac{\partial V}{\partial r}\big|_{r_0}}\right|$$

$$\le (\text{const})\left|\frac{\partial V}{\partial r}\Big|_{r_0}\right|$$

which is finite

EMII-2



p_x: $E = E'\cos\theta + p'\cos\theta_e$ 3 unknowns: θ_e, p', E'

p_y: $0 = E'\sin\theta + p'\sin\theta_e \to p'^2 = (E-E'\cos\theta)^2 + E'^2\sin^2\theta = E^2+E'^2-2EE'\cos\theta$

E: $E+m = E' + \sqrt{p'^2+m^2} \to E+m = E' + \sqrt{E^2+E'^2+m^2-2EE'\cos\theta}$

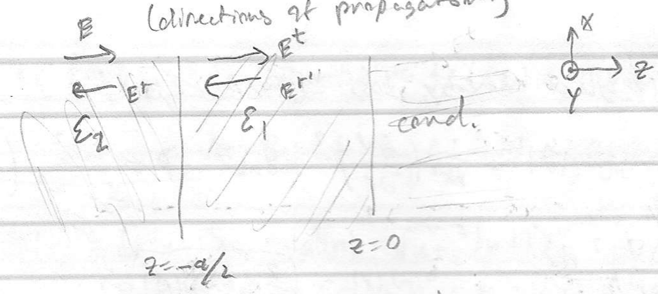
$(E+m-E')^2 = \cancel{E^2}+\cancel{m^2}+\cancel{E'^2}+2Em-2EE'-2mE' = \cancel{E^2}+\cancel{E'^2}+\cancel{m^2}-2EE'\cos\theta$

$\Rightarrow 2Em = E'(2E+2m-2E\cos\theta)$

$\Rightarrow E' = \frac{Em}{m+E(1-\cos\theta)} = \frac{1}{\frac{1}{E}+\frac{1}{m}(1-\cos\theta)}$ (taking $c=1$)

$\gamma = \frac{E+m-E'}{m} = 1+\frac{E}{m} - \frac{1}{\frac{m}{E}+1-\cos\theta}$

2012 EMII-3



E points in $\hat{y}$ direction

$$z<-a/2: \quad E = E_0 e^{ik_2(z+a/2)-i\omega t} + E_0^r e^{-ik_2(z+a/2)-i\omega t}, \quad k_2 = \omega\sqrt{\mu_0\epsilon_2}$$

$$-a/2<z<0: \quad E' = E^t e^{ik_1 z - i\omega t} + E^{r'} e^{-ik_1 z - i\omega t}, \quad k_1 = \omega\sqrt{\mu_0\epsilon_1}$$

$$E'\big|_{z=0} = 0 \Rightarrow E^t e^{-i\omega t} + E^{r'} e^{-i\omega t} = 0 \Rightarrow E^{r'} = -E^t$$

$$E\big|_{z=-a/2} = E'\big|_{z=-a/2} \Rightarrow E_0 + E_0^r = E_0^t\left(e^{-ik_1 a/2} - e^{ik_1 a/2}\right) = -2i E^t \sin(k_1 a/2)$$

$$\vec{B} = \frac{1}{c}\hat{k}\times\vec{E} = \sqrt{\mu_0\epsilon}\,\hat{k}\times\vec{E}, \quad \vec{H} = \frac{\vec{B}}{\mu_0}$$

$$H\big|_{z=-a/2} = -\sqrt{\frac{\epsilon_2}{\mu_0}}\left(E_0 - E_0^r\right)e^{-i\omega t}$$

$$H'\big|_{z=-a/2} = -\sqrt{\frac{\epsilon_1}{\mu_0}}\left(E_0^t e^{-ik_1 a/2} - E_0^{r'} e^{ik_1 a/2}\right)e^{-i\omega t}$$

$$\Rightarrow \sqrt{\epsilon_2}\left(E_0 - E_0^r\right) = \sqrt{\epsilon_1}\,E_0^t \cdot 2\cos(k_1 a/2)$$

$$E_0 - E_0^r = \sqrt{\frac{\epsilon_1}{\epsilon_2}}\,2E_0^t\cos(k_1 a/2)$$

$$\Rightarrow E_0 = E_0^t\left(\sqrt{\frac{\epsilon_1}{\epsilon_2}}\cos(k_1 a/2) - i\sin(k_1 a/2)\right) \Rightarrow E_0^t = \frac{E_0}{\sqrt{\frac{\epsilon_1}{\epsilon_2}}\cos(k_1 a/2) - i\sin(k_1 a/2)}$$

$$E_0^r = -E_0^t\left(\sqrt{\frac{\epsilon_1}{\epsilon_2}}\cos(k_1 a/2) + i\sin(k_1 a/2)\right)$$

$$= -E_0\,\frac{\sqrt{\epsilon_1/\epsilon_2}\cos(k_1 a/2) + i\sin(k_1 a/2)}{\sqrt{\epsilon_1/\epsilon_2}\cos(k_1 a/2) - i\sin(k_1 a/2)}$$

note $\sqrt{\epsilon_1/\epsilon_2} = \frac{k_1}{k_2}$

$$r_a = \frac{E_0^r}{E_0} = -\frac{k_1\cos(k_1 a/2) + ik_2\sin(k_1 a/2)}{k_1\cos(k_1 a/2) - ik_2\sin(k_1 a/2)}$$

with $k_i = \omega\sqrt{\mu_0\epsilon_i}$

$$|r_a|^2 = \frac{k_1^2\cos^2(k_1 a/2) + k_2^2\sin^2(k_1 a/2)}{k_1^2\cos^2(k_1 a/2) + k_2^2\sin^2(k_1 a/2)} = 1$$

All radiation reflects back from conductor.

The dielectric arrangement only changes the phase.

EMII-4 $\nabla\cdot\vec{E} = 4\pi\rho \quad \nabla\cdot\vec{B} = 0 \quad \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \quad \nabla\times\vec{B} = 4\pi\vec{j} + \frac{\partial\vec{E}}{\partial t}$ (take $c=1$)

$$\vec{E} = -\nabla\phi - \frac{\partial\vec{A}}{\partial t} \qquad \vec{B} = \nabla\times\vec{A}$$

$$\nabla\cdot\vec{E} = -\nabla^2\phi - \frac{\partial}{\partial t}\nabla\cdot\vec{A}\,, \quad \nabla\times\vec{B} = \nabla\times(\nabla\times\vec{A}) = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A} = 4\pi\vec{j} + \frac{\partial}{\partial t}\left(-\nabla\phi - \frac{\partial\vec{A}}{\partial t}\right)$$

$$\Rightarrow \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A} + \nabla\left(\frac{\partial\phi}{\partial t}\right) + \frac{\partial^2\vec{A}}{\partial t^2} = 4\pi\vec{j}$$

$$\left(-\nabla^2 + \partial_t^2\right)\vec{A} + \nabla\left(\nabla\cdot\vec{A} + \frac{\partial\phi}{\partial t}\right) = 4\pi\vec{j}$$

$$-\nabla^2\phi - \frac{\partial}{\partial t}\nabla\cdot\vec{A} = 4\pi\rho\,, \quad \text{take } \nabla\cdot\vec{A} = -\frac{\partial\phi}{\partial t} \Rightarrow -\nabla^2\phi + \frac{\partial^2}{\partial t^2}\phi = 4\pi\rho$$

$$\Rightarrow -\square\vec{A} = 4\pi\vec{j}\,, \quad -\square\phi = 4\pi\rho$$

$$\phi(\vec{r},t) = \int dt'\int d^3r'\,\rho(\vec{r}',t')\,G(\vec{r}-\vec{r}',t-t')$$

since $-\square\phi(\vec{r},t) = \int dt'\int d^3r'\,\rho(\vec{r}',t')\underbrace{\left[-\square G(\vec{r}-\vec{r}',t-t')\right]}_{4\pi\delta(\vec{r}-\vec{r}',t-t')} = 4\pi\rho(\vec{r},t)$

likewise $\vec{A}(\vec{r},t) = \int dt'\int d^3r'\,\vec{j}(\vec{r}',t')\,G(\vec{r}-\vec{r}',t-t')$

Use + sign for causality ($t > t'$)

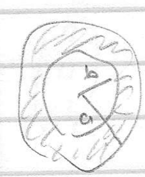
$$\phi(\vec{r},t) = \int dt'\,d^3r'\,\rho(\vec{r}',t')\frac{1}{|\vec{r}-\vec{r}'|}\delta\left(|\vec{r}-\vec{r}'| - (t-t')\right) \qquad (c=1)$$

$\to t' = t - |\vec{r}-\vec{r}'|$

$$= \int d^3r'\,\frac{\rho(\vec{r}',t-|\vec{r}-\vec{r}'|)}{|\vec{r}-\vec{r}'|}$$

$$A(\vec{r},t) = \int d^3r'\,\frac{\vec{j}(\vec{r}',t-|\vec{r}-\vec{r}'|)}{|\vec{r}-\vec{r}'|}$$

EMII-5 TM: $(\nabla_t^2 + \gamma^2)E_z = 0 \quad (B_z = 0)\,, \quad \gamma^2 = \mu\epsilon\omega^2 - k^2$



$$\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2} + \gamma^2\right)E_z = 0\,, \quad E_z\big|_{\rho=a} = E_z\big|_{\rho=b} = 0$$

$$E_z = \Phi(\rho)\chi(\varphi) \Rightarrow \frac{\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}\Phi}{\Phi} + \frac{1}{\rho^2}\frac{\chi''}{\chi} + \gamma^2 = 0$$

$\frac{\chi''}{\chi} \equiv -m^2$ constant in ρ $\Rightarrow \chi(\varphi) \propto e^{\pm im\varphi}\,, \quad m\in\mathbb{Z}$ (since $\chi(\varphi+2\pi) = \chi(\varphi)$)

and $\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} - \frac{m^2}{\rho^2} + \gamma^2\right)\Phi = 0$

2012 $\Rightarrow \left(\frac{\partial^2}{\partial\rho^2} - \frac{m^2}{\rho^2} + \frac{1}{4}\frac{1}{\rho^2} + \gamma^2\right)\underbrace{\sqrt{\rho}\,\Phi(\rho)}_{\Psi(\rho)} = 0 \quad , \quad \Phi(a) = \Phi(b) = 0$

$$\Psi'' - \frac{(m^2 - 1/4)}{\rho^2}\Psi + \gamma^2\Psi \simeq \Psi'' + \left[\gamma^2 - \frac{m^2 - 1/4}{a^2}\right]\Psi = 0$$

since $a - b \ll a \Rightarrow |\rho - a| \ll \rho$

$$\Psi(\rho) \propto e^{\pm i\sqrt{\gamma^2 - \frac{m^2 - 1/4}{a^2}}\,\rho}$$

insert constant factor

$$\Phi(\rho) \propto \frac{1}{\sqrt{\rho}} e^{\pm i\sqrt{\gamma^2 - \frac{m^2-1/4}{a^2}}(\rho - b)} \quad , \quad \text{but } \Phi(a) = \Phi(b) = 0$$

$$\Phi(\rho) = \frac{A}{\sqrt{\rho}}\cos\left[\sqrt{\gamma^2 - \frac{m^2-1/4}{a^2}}(\rho - b)\right] + \frac{B}{\sqrt{\rho}}\sin\left[\sqrt{\gamma^2 - \frac{m^2-1/4}{a^2}}(\rho - b)\right]$$

$$\Phi(b) = 0 \Rightarrow A = 0 \ , \ \Phi(a) = 0 \Rightarrow \sqrt{\gamma^2 - \frac{m^2-1/4}{a^2}}(a - b) = n\pi \ , \ n \in \mathbb{Z}$$

$$\Rightarrow \gamma^2 = \frac{m^2 - 1/4}{a^2} + \frac{n^2\pi^2}{(a-b)^2} \ , \ (m, n) \in \mathbb{Z}^2$$

$$\Rightarrow \gamma_{mn} = \sqrt{\frac{m^2 - 1/4}{a^2} + \frac{n^2\pi^2}{(a-b)^2}} \ , \ (m,n) \in \mathbb{Z}^2 \setminus (0,0)$$

QMII-1 $\underbrace{\Theta}_{\text{t-reversal}} H = H\Theta$, so $H\Theta|n\rangle = \Theta H|n\rangle = E_n \Theta|n\rangle$

$\Rightarrow |n\rangle, \Theta|n\rangle$ are same state (nondegenerate)

but $|n\rangle = \int dx\, |x\rangle\langle x|n\rangle$, $\Theta|n\rangle = \int dx\, \Theta|x\rangle\langle x|n\rangle^* = \int dx\, |x\rangle\langle x|n\rangle^*$

$\Rightarrow \langle x|n\rangle = \langle x|n\rangle^* e^{i\theta}$

const phase factor, can take to be 1 in which case $\langle x|n\rangle$ is real.

plane wave is degenerate, e.g. $e^{ip x}$ and e^{-ipx} have same energy.

QMII-2 $|n, l=1, m=\pm1, 0\rangle \to |1\rangle, |-1\rangle, |0\rangle$

$\langle \vec{x}|1\rangle \propto \sin\theta\, e^{i\varphi} \propto x+iy \qquad \langle\vec{x}|0\rangle \propto \cos\theta \propto z$

$\langle\vec{x}|-1\rangle \propto \sin\theta\, e^{-i\varphi}\propto x-iy$

$V=\lambda(x^2-y^2)=\lambda r^2\sin^2\theta(\cos^2\varphi-\sin^2\varphi)$

Consider angular parts: $\langle\Omega|1\rangle = \frac{A}{\sqrt2}\sin\theta e^{i\varphi}$, $\langle\Omega|-1\rangle=\frac{A}{\sqrt2}\sin\theta e^{-i\varphi}$,

$\langle\Omega|0\rangle = A\cos\theta$

$$\langle i|V|j\rangle = \lambda \underset{\text{radial integral}}{R} \int d\Omega\, \langle i|\Omega\rangle \sin^2\theta(\cos^2\varphi-\sin^2\varphi)\langle\Omega|j\rangle$$

$$\langle 1|V|1\rangle = \lambda RA^2\frac12\int\sin\theta\, d\theta\, d\varphi\, \sin^2\theta(\cos^2\varphi-\sin^2\varphi)\sin\theta e^{-i\varphi}\sin\theta e^{i\varphi}$$

$=0$ since $\int\cos^2\varphi\, d\varphi = \int\sin^2\varphi\, d\varphi$

$\langle -1|V|-1\rangle = 0$ similarly

$$\langle\pm1|V|\mp1\rangle = \lambda RA^2 \frac12 \underbrace{\int \sin^5\theta\, d\theta}\ \underbrace{\int d\varphi\, (\underbrace{\cos^2\varphi-\sin^2\varphi}_{\cos(2\varphi)})\, e^{\mp2i\varphi}}_{\pi}$$

$e^{\mp 2i\varphi}$: $\cos(2\varphi)\pm i\sin(2\varphi)$

$$\int_{-1}^{1} d\mu\,(1-\mu^2)^2 = \int_{-1}^{1} d\mu\,(1-2\mu^2+\mu^4) = 2-\frac43+\frac25=\frac{16}{15}$$

$$=\lambda RA^2\,\frac{8\pi}{15}$$

$$\langle0|V|0\rangle = \lambda RA^2\int\sin\theta\, d\theta\, d\varphi\,\cos^2\theta\,\sin^2\theta\,\overbrace{(\cos^2\varphi-\sin^2\varphi)}^{\text{cancel}} = 0$$

$$\langle 0|V|\pm1\rangle = \lambda RA^2\frac1{\sqrt2}\int\sin\theta\, d\theta\, d\varphi\, \sin^2\theta(\cos^2\varphi-\sin^2\varphi)\cos\theta\sin\theta e^{\pm i\varphi}$$

$$=\lambda RA^2\frac{1}{\sqrt2}\underbrace{\int_0^\pi d\sin\theta\,\sin^4\theta}_{\frac15\sin^5\theta|_0^\pi=0}\int d\varphi(\cos^2\varphi-\sin^2\varphi)(\cos\varphi\pm i\sin\varphi)=0$$

$\langle\pm1|V|0\rangle=0$ similarly

$$V=\lambda RA^2\ \begin{array}{c|c} +1 & \\ -1 & \\ 0 & \end{array}\begin{pmatrix} 0 & \frac{8\pi}{15} & 0\\ \frac{8\pi}{15} & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \to \text{eigenstates } \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$\Rightarrow \quad \frac1{\sqrt2}(|1\rangle+|-1\rangle), \quad \frac1{\sqrt2}(|1\rangle-|-1\rangle), \quad |0\rangle$

w/ eigenvalues $\frac{8\pi}{15}\lambda RA^2$, $-\frac{8\pi}{15}\lambda RA^2$, 0

note $\Theta|m\rangle \sim \Theta Y_l^m = Y_l^{m*} = (-1)^m Y_l^{-m}\sim(-1)^m|-m\rangle$, so

$\Theta\frac1{\sqrt2}(|1\rangle+|-1\rangle) = \frac{-1}{\sqrt2}(|1\rangle+|-1\rangle)$, $\Theta\frac{1}{\sqrt2}(|1\rangle-|-1\rangle)=\frac1{\sqrt2}(|1\rangle-|-1\rangle)$, $\Theta|0\rangle=|0\rangle$

QM11-3 $V = e^2 A^2 / 2mc^2$, $\vec{A} = \frac{1}{2}\vec{B}\times\vec{r}$ (so $\nabla \times \vec{A} = \vec{B}$)

$$V = \frac{e^2}{8mc^2}(\vec{B}\times\vec{r})^2 , \quad \text{set } \vec{B} = B\hat{z}$$

$$\vec{B}\times\vec{r} = B\hat{z}\times(r\cos\theta\,\hat{z} + r\sin\theta\cos\varphi\,\hat{x} + r\sin\theta\sin\varphi\,\hat{y})$$

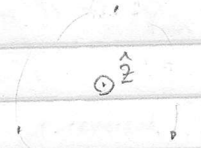
$$= B(-r\sin\theta\sin\varphi\,\hat{x} + r\sin\theta\cos\varphi\,\hat{y}) = Br\sin\theta(-\sin\varphi\,\hat{x} + \cos\varphi\,\hat{y})$$

$$V = \frac{e^2}{8mc^2}B^2 r^2 \sin^2\theta$$

$$\langle 0|V|0\rangle = \frac{1}{\pi a_0^3}\frac{e^2}{8mc^2}B^2 \underbrace{\int_0^\infty r^2 dr\, e^{-2r/a_0} r^2}_{\frac{4!}{(\frac{2}{a})^5} = \frac{24}{32}a_0^5} \underbrace{\int_0^\pi 2\pi\sin\theta\, d\theta\, \sin^2\theta}_{2\pi\int_{-1}^{1}(1-\mu^2)\,d\mu = 2\pi\left(2-\frac{2}{3}\right) = \frac{8\pi}{3}}$$

$$= \frac{1}{\pi a_0^3}\frac{e^2}{8mc^2}B^2\frac{3}{4}a_0^5\frac{8\pi}{3} = \frac{e^2 B^2 a_0^2}{4mc^2}$$

QM11-4



must be symmetric under $\frac{\pi}{3}$ rotation.

but $Y_l^m \propto e^{im\varphi}$, so $e^{im\pi/3} = 1 \Rightarrow \frac{m\pi}{3} = 2\pi n$

$$\Rightarrow m = 6n, \quad n \in \mathbb{Z}$$

QM11-5 $H = -\frac{\hbar^2}{2mr^2}\partial_r r^2 \partial_r + \frac{l(l+1)\hbar^2}{2mr^2} + g\,\delta(r-r_0)$, integrate $H\psi = E\psi$ across shell.

$$\int_{r_0-\epsilon}^{r_0+\epsilon} dr\, \frac{-\hbar^2}{2m}\partial_r r^2 \psi(r) + r_0^2 g\,\psi(r_0) = 0 \Rightarrow \underbrace{\frac{\partial}{\partial r}(r^2\psi(r))}_{r^2\psi' + 2r\psi}\Big|_{r_0-\epsilon}^{r_0+\epsilon} = \frac{2mg}{\hbar^2} r_0^2 \psi(r_0)$$

$$\Rightarrow \frac{\partial\psi}{\partial r}\Big|_{r_0-\epsilon}^{r_0+\epsilon} = \frac{2mg}{\hbar^2}\psi(r_0) \qquad \Rightarrow \frac{\partial}{\partial r}(r\psi)\Big|_{r_0-\epsilon}^{r_0+\epsilon} = \frac{2mg}{\hbar^2}(r\psi)\Big|_{r=r_0}$$

Also $\psi(r_0+\epsilon) = \psi(r_0-\epsilon)$

$\frac{\partial\psi}{\partial r}\Big|_{r=0} = 0$ for continuity of derivative and $r\psi|_{r=0} = 0$

find solutions to $-\frac{\hbar^2}{2mr^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}\psi = E\psi = \frac{\hbar^2 k^2}{2m}\psi$

Take $\eta = r\psi \Rightarrow \frac{\partial\psi}{\partial r} = \frac{1}{r}\frac{\partial\eta}{\partial r} - \frac{\eta}{r^2} \Rightarrow \frac{\partial}{\partial r}r^2\frac{\partial\psi}{\partial r} = \frac{\partial\eta}{\partial r} + r\frac{\partial^2\eta}{\partial r^2} - \frac{\partial\eta}{\partial r} = r\frac{\partial^2\eta}{\partial r^2}$

$\Rightarrow \frac{\partial^2}{\partial r^2}(r\psi) = -k^2 r\psi \Rightarrow r\psi \propto \frac{1}{k}\sin(kr) + f_0(k)\,e^{ikr}$ outside r_0

incoming plane wave, s component

$r > r_0$: $r\psi = A\left[\frac{\sin kr}{k} + f_0(k)e^{ikr}\right]$

$r < r_0$: $r\psi = B\sin kr$ (so $r\psi|_{r=0} = 0$)

$$A\left[\cos kr_0 + ikf_0(k)e^{ikr_0}\right] - kB\cos kr_0 = \frac{2mg}{\hbar^2}B\sin kr_0$$

$$A\left[\frac{\sin kr_0}{k} + f_0(k)e^{ikr_0}\right] = B\sin kr_0$$

$$A\left[\cos kr_0 + ikf_0(k)e^{ikr_0}\right] = \frac{A\left[\frac{\sin kr_0}{k} + f_0(k)e^{ikr_0}\right]}{\sin kr_0}\left(k\cos kr_0 + \frac{2mg}{\hbar^2}\sin kr_0\right)$$

$$f_0(k)\left[ike^{ikr_0} - \frac{e^{ikr_0}}{\sin kr_0}\left(k\cos kr_0 + \frac{2mg}{\hbar^2}\sin kr_0\right)\right] = -\cos kr_0 + \frac{1}{k}\left(k\cos kr_0 + \frac{2mg}{\hbar^2}\sin kr_0\right)$$

$$f_0(k)\ ke^{ikr_0}\left(i - \left(\cot kr_0 - \frac{2mg}{\hbar^2 k}\right)\right) = \frac{2mg}{\hbar^2 k}\sin kr_0$$

$$f_0(k) = \frac{-\sin kr_0}{ke^{ikr_0}}\ \frac{1}{1 + \frac{\hbar^2 k}{2mg}(\cot kr_0 - i)}$$

bound states $(l=0)$: $\frac{\partial^2}{\partial r^2}(r\psi) = -k^2 r\psi$ for infinite spherical well

$r\psi = A\sin(kr)$ to satisfy $r\psi|_{r=0} = 0$

but $r\psi|_{r=r_0} = 0 \Rightarrow A\sin(kr_0) = 0 \Rightarrow kr_0 = n\pi$, $n \in \mathbb{Z}\setminus\{0\}$

$$k = \frac{n\pi}{r_0}$$

∴ resonances will be similar to these bound states, except with leakage outside the barrier.

$$f_0\left(\frac{n\pi}{r_0}\right) \cong -\frac{1}{k}e^{-in\pi}\ \frac{\overbrace{\sin(n\pi)}^{\to 0}}{\frac{\hbar^2 k}{2mg}\underbrace{\cot(n\pi)}_{\to\infty}} = 0$$