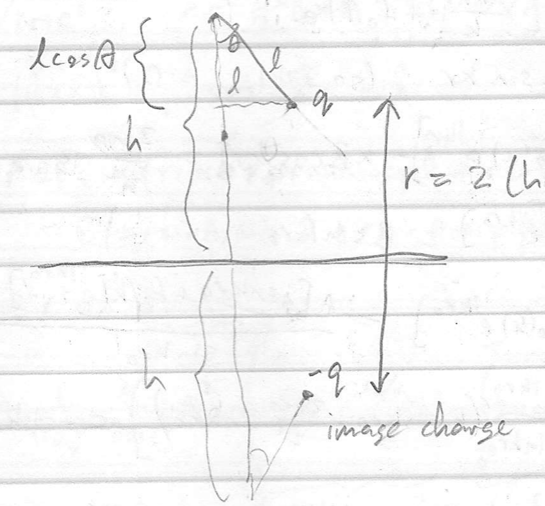
2011

CM1



$T = \frac{1}{2} m l^2 \dot{\theta}^2$

$r = 2(h - l\cos\theta)$

$V = \frac{-kq^2}{r}$

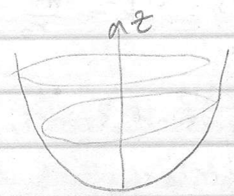
$$L = \frac{1}{2} m l^2 \dot{\theta}^2 + \frac{kq^2}{2(h - l\cos\theta)} \quad , \quad k = \frac{1}{4\pi\epsilon_0}$$

$$m l^2 \ddot{\theta} + \frac{kq^2 l \sin\theta}{2(h - l\cos\theta)^2} = 0 \quad , \quad \text{small } \theta \Rightarrow \sin\theta \simeq \theta, \cos\theta \simeq 1$$

$$m l^2 \ddot{\theta} + \frac{kq^2 l}{2h}\theta = 0 \Rightarrow \ddot{\theta} + \frac{kq^2}{2hlm}\theta = 0$$

$$\omega = \sqrt{\frac{kq^2}{2hlm}}$$

CM2



$z = \rho^2$. $\quad V = mgz = mg\rho^2$

$\dot{z} = 2\rho\dot{\rho} \quad T = \frac{1}{2} m(\dot{z}^2 + \dot{\rho}^2 + \rho^2\dot{\theta}^2)$

$\quad = \frac{1}{2} m\left((4\rho^2+1)\dot{\rho}^2 + \rho^2\dot{\theta}^2\right)$

$$L = \frac{1}{2} m\left((4\rho^2+1)\dot{\rho}^2 + \rho^2\dot{\theta}^2\right) - mg\rho^2$$

$$\frac{\partial L}{\partial \dot{\theta}} = m\rho^2\dot{\theta} \equiv l \text{ constant} \qquad \dot{\theta} = \frac{l}{m\rho^2}$$

$$\frac{d}{dt}\left[m(4\rho^2+1)\dot{\rho}\right] - m(4\rho\dot{\rho}^2 + \rho\dot{\theta}^2) + 2mg\rho = 0$$

$$m(8\rho\dot{\rho}^2 + (4\rho^2+1)\ddot{\rho}) - 4m\rho\dot{\rho}^2 - \frac{l^2}{m\rho^3} + 2mg\rho = 0$$

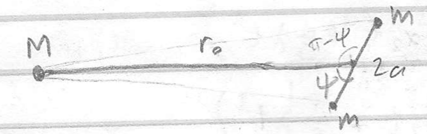
$$(4\rho^2+1)\ddot{\rho} + 4\rho\dot{\rho}^2 - \frac{l^2}{m\rho^3} + 2mg\rho = 0$$

$z = 1 \Rightarrow \rho = 1$, circular $\Rightarrow \ddot{\rho} = \dot{\rho} = 0 \Rightarrow -\frac{l^2}{m} + 2mg = 0$

$\quad l^2 = 2m^2 g \Rightarrow l = \sqrt{2g}\, m$

$$(1+x)^{-1/2} = 1 - \frac{x}{2} + \frac{3}{8}x^2 + \ldots$$

CM3



$$T_{rot} = ma^2(\dot\psi + \Omega)^2, \quad \Omega^2 = GM/r_0^3$$

$$V = -GMm\left[(r_0^2+a^2+2ar_0\cos\psi)^{-1/2} + (r_0^2+a^2-2ar_0\cos\psi)^{-1/2}\right]$$

$$= -\frac{GMm}{r_0}\left[\left(1+2\frac{a}{r_0}\cos\psi + \left(\frac{a}{r_0}\right)^2\right)^{-1/2} + \left(1-2\frac{a}{r_0}\cos\psi + \left(\frac{a}{r_0}\right)^2\right)^{-1/2}\right]$$

$$\simeq -\frac{GMm}{r_0}\left[1 - \cancel{\frac{a}{r_0}\cos\psi} - \frac{1}{2}\left(\frac{a}{r_0}\right)^2 + \frac{3}{8}4\left(\frac{a}{r_0}\right)^2\cos^2\psi + 1 + \cancel{\frac{a}{r_0}\cos\psi} - \frac{1}{2}\left(\frac{a}{r_0}\right)^2 + \frac{3}{8}4\cos^2\psi\left(\frac{a}{r_0}\right)^2\right]$$

$$\simeq -\frac{GMm}{r_0}\left[2 + (-1+3\cos^2\psi)\left(\frac{a}{r_0}\right)^2\right]$$

$$\Rightarrow V_{rot} \simeq \frac{GMm a^2}{r_0^3}(1-3\cos^2\psi)$$

$$L = ma^2(\dot\psi+\Omega)^2 - \frac{GMma^2}{r_0^3}(1-3\cos^2\psi), \quad \Omega^2 = GM/r_0^3$$

$$2ma^2\ddot\psi + \frac{6GMma^2}{r_0^3}\cos\psi\sin\psi = 0$$

$$\frac{\partial V_{rot}}{\partial\psi} = \frac{6GMma^2}{r_0^3}\cos\psi\sin\psi = 0 \Rightarrow \cos\psi\sin\psi = 0 \Rightarrow \sin(2\psi) = 0$$

$\Rightarrow \psi = 0, \pi/2, \pi, \frac{3\pi}{2}$ are equilibrium points

$$\frac{\partial^2 V_{rot}}{\partial\psi^2} = \frac{\partial}{\partial\psi}\frac{3GMma^2}{r_0^3}\sin(2\psi) = \frac{6GMma^2}{r_0^3}\cos(2\psi)$$

>0 at $\psi = 0$ and $\psi = \pi$ (stable)

<0 at $\psi = \pi/2$ and $\psi = 3\pi/2$ (unstable)

CM4 $H = \frac{p^2}{2m} - \lambda x t$, $\quad p = \frac{\partial S}{\partial x} \rightarrow H\left(x, \frac{\partial S}{\partial x}\right) + \frac{\partial S}{\partial t} = 0$

$$\frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 - \lambda x t + \frac{\partial S}{\partial t} = 0 \Rightarrow S = S_x(x) + \frac{1}{2}\lambda x t^2 + S_t(t) + \underset{\text{const of integ.}}{\alpha_1}$$

$$\frac{1}{2m}\left(\frac{\partial S_x}{\partial x} + \frac{1}{2}\lambda t^2\right)^2 + \frac{\partial S_t}{\partial t} = 0$$

take $S_x = \alpha_2 x$, $\quad \frac{1}{2m}\left(\alpha_2 + \frac{1}{2}\lambda t^2\right)^2 + \frac{\partial S_t}{\partial t} = 0$

$$\frac{\partial S_t}{\partial t} = \frac{-1}{2m}\left(\alpha_2^2 + \lambda\alpha_2 t^2 + \frac{1}{4}\lambda^2 t^4\right) \Rightarrow S_t = \frac{-1}{2m}\left(\alpha_2^2 t + \frac{1}{3}\lambda\alpha_2 t^3 + \frac{1}{20}\lambda^2 t^5\right)$$

$$S = \alpha_2 x + \frac{1}{2}\lambda x t^2 - \frac{1}{2m}\left(\alpha_2^2 t + \frac{1}{3}\lambda\alpha_2 t^3 + \frac{1}{20}\lambda^2 t^5\right) + \alpha_1$$

$$\beta = \frac{\partial S}{\partial \alpha_2} = x - \frac{1}{2m}\left(2\alpha_2 t + \frac{1}{3}\lambda t^3\right)$$

$$\Rightarrow x = \beta + \frac{1}{2m}\left(2\alpha_2 t + \frac{1}{3}\lambda t^3\right), \quad x_0 = x(0) = \beta$$

$$p = \frac{\partial S}{\partial x} = \alpha_2 + \frac{1}{2}\lambda t^2, \quad p_0 = p(0) = \alpha_2$$

So $x = x_0 + \frac{p_0}{m} t + \frac{\lambda}{6m} t^3$.

CM 5 $[A,H] = \frac{\partial A}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial H}{\partial q}\frac{\partial A}{\partial p} = \frac{\partial A}{\partial q}\dot{q} + \frac{\partial A}{\partial p}\dot{p} = \frac{dA}{dt}$

Now suppose $\underbrace{[[\dots[A,H],H],\dots,H]}_{n \text{ times}} = \frac{d^n A}{dt^n}$

then $\underbrace{[[[\dots[A,H],H],\dots,H]}_{n+1 \text{ times}} = \left[\frac{d^n A}{dt^n}, H\right] = \frac{d^{n+1}A}{dt^{n+1}}$

$\therefore$ by induction, $[[\dots[A,H],H]\dots,H] = \frac{d^n A}{dt^n} \quad \forall n \in \mathbb{N}$

$$q(t) = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{d^n q}{dt^n}\bigg|_{t=0} t^n \quad \text{(Taylor expansion about } t=0\text{)}$$

$$= q(0) + \frac{dq}{dt}\bigg|_{t=0} t + \frac{1}{2}\frac{d^2q}{dt^2}t^2 + \frac{1}{6}\frac{d^3q}{dt^2}t^3 + \dots$$

$$= q(0) + [q,H]\big|_{t=0} t + \frac{1}{2}[[q,H],H]\big|_{t=0} t^2 + \frac{1}{6}[[[q,H],H],H]\big|_{t=0} t^3 + \dots$$

$H = \frac{p^2}{2m} + \frac{1}{2}kq^2$, $[q,H] = \frac{\partial H}{\partial p} = \frac{p}{m}$, $[[q,H],H] = \frac{1}{m}[p,H] = -\frac{kq}{m}$

$[[[q,H],H],H] = -\frac{k}{m}[q,H] = -\frac{kp}{m^2}$, $[\dots 4 \dots] = -\frac{k}{m^2}[p,H] = \frac{k^2}{m^2} q$,

$$\underbrace{[[\dots[q,H],H],\dots,H]}_{n} = \begin{cases} (-1)^{n/2}\left(\frac{k}{m}\right)^{n/2} q, & n \text{ even} \\ (-1)^{\frac{n-1}{2}}\left(\frac{k}{m}\right)^{\frac{n-1}{2}} \frac{p}{m}, & n \text{ odd} \end{cases}$$

$$q(t) = q_0 + \frac{p_0}{m}t - \frac{1}{2}\frac{k}{m}q_0 t^2 - \frac{1}{6}\frac{k}{m}\frac{p_0}{m}t^3 + \frac{1}{4!}\left(\frac{k}{m}\right)^2 q_0 t^4 + \frac{1}{5!}\left(\frac{k}{m}\right)^2 \frac{p_0}{m}t^5 + \dots$$

$$= q_0\left(1 - \frac{1}{2}\frac{k}{m}t^2 + \frac{1}{4!}\left(\frac{k}{m}\right)^2 t^4 + \dots\right) + \frac{p_0}{m}\left(t - \frac{1}{3!}\frac{k}{m}t^3 + \frac{1}{5!}\left(\frac{k}{m}\right)^2 t^5 + \dots\right)$$

$$= q_0 \cos\left(\sqrt{\frac{k}{m}}t\right) + \frac{p_0}{m}\sqrt{\frac{m}{k}}\sin\left(\sqrt{\frac{k}{m}}t\right)$$

(note: distribution function, not density of states)

SM1 $Q_i = \sum_{n_i=0}^{\infty} e^{-\beta(n_i+\frac{1}{2})\hbar\omega_i} = \frac{e^{-\beta\hbar\omega_i/2}}{1-e^{-\beta\hbar\omega_i}}, \quad A_i = -kT\left(-\frac{\beta\hbar\omega_i}{2} - \log(1-e^{-\beta\hbar\omega_i})\right)$

$$S_i = -\frac{\partial A_i}{\partial T} = k\left(\cancel{-\frac{\beta\hbar\omega_i}{2}} - \log(1-e^{-\beta\hbar\omega_i})\right) + kT\left(\cancel{\frac{\hbar\omega_i}{2kT^2}} + \frac{\frac{\hbar\omega_i}{kT^2}e^{-\beta\hbar\omega_i}}{1-e^{-\beta\hbar\omega_i}}\right)$$

$$= \frac{1}{T}\frac{\hbar\omega_i}{e^{\beta\hbar\omega_i}-1} - k\log(1-e^{-\hbar\omega_i/kT})$$

$$S = \int_0^\infty S_i(\omega)g(\omega)d\omega = \frac{1}{T}\int_0^\infty \frac{\hbar\omega}{e^{\hbar\omega/kT}-1}g(\omega)d\omega - k\int_0^\infty \log(1-e^{-\hbar\omega/kT})g(\omega)d\omega$$

SM2 $Q = \int_{-1}^{1} 2\pi\, d\cos\theta\, e^{-\beta\frac{qp\cos\theta}{4\pi\epsilon_0}\frac{1}{r^2}} = \frac{-2\pi\cdot 4\pi\epsilon_0 r^2}{\beta q p}\left(e^{-\frac{\beta qp}{4\pi\epsilon_0 r^2}} - e^{\frac{\beta qp}{4\pi\epsilon_0 r^2}}\right)$

$$u = -\frac{\partial\log Q}{\partial\beta} = -\frac{\partial}{\partial\beta}\left(-\log\beta + \log\left(e^{\beta qp/4\pi\epsilon_0 r^2} - e^{-\beta qp/4\pi\epsilon_0 r^2}\right)\right)$$

$$= -\left(-\frac{1}{\beta} + \frac{\frac{qp}{4\pi\epsilon_0 r^2}\left(e^{\beta qp/4\pi\epsilon_0 r^2} + e^{-\beta qp/4\pi\epsilon_0 r^2}\right)}{e^{\beta qp/4\pi\epsilon_0 r^2} - e^{-\beta qp/4\pi\epsilon_0 r^2}}\right) = kT - \frac{qp}{4\pi\epsilon_0 r^2}\coth\left(\frac{\beta qp}{4\pi\epsilon_0 r^2}\right)$$

but $\coth x = \frac{1+\frac{x^2}{2}+\cdots}{x+\frac{x^3}{3!}+\cdots} \simeq \frac{1}{x}\left(1+\frac{x^2}{2}-\frac{x^2}{6}\right) = \frac{1}{x}+\frac{x}{3}$

$$u \simeq \frac{1}{\beta} - \left(\frac{1}{\beta} + \frac{\beta}{3}\left(\frac{qp}{4\pi\epsilon_0 r^2}\right)^2\right) = \frac{1}{3kT}\left(\frac{qp}{4\pi\epsilon_0}\right)^2\frac{1}{r^4}$$

SM3 $Q_N = \sum_{\substack{\{n_\varepsilon = 0 \text{ or } 1\} \\ \{\sum_\varepsilon n_\varepsilon = N\}}} e^{-\beta\sum_\varepsilon n_\varepsilon \varepsilon}, \quad \varepsilon = \frac{p^2}{2m}, \quad \Sigma(p) = \frac{4}{3}\pi p^3\cdot\frac{V}{h^3}$

$$\frac{d\Sigma}{dp} = 4\pi p^2\frac{V}{h^3}$$

$$Q_2 = \sum_{\substack{\{n_\varepsilon=0\text{ or }1\}\\ \{\sum n_\varepsilon = 2\}}}\prod_\varepsilon e^{-\beta n_\varepsilon\varepsilon} = \underbrace{\sum_{\substack{\{n_\varepsilon\in\mathbb{N}\}\\ \{\sum n_\varepsilon = 2\}}}\prod_\varepsilon e^{-\beta n_\varepsilon\varepsilon}}_{Q_2^{(\text{classical})} = \frac{1}{2!}\left(\frac{V}{\lambda^3}\right)^2} - \sum_{\substack{\{\text{some } n_\varepsilon > 1\}\\ \{\sum n_\varepsilon = 2\}}}\prod_\varepsilon e^{-\beta n_\varepsilon\varepsilon}$$

only a single state twice occupied

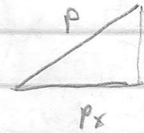
$$= \frac{1}{2!}\left(\frac{V}{\lambda^3}\right)^2 - \sum_\varepsilon e^{-2\beta\varepsilon} = \frac{1}{2!}\left(\frac{V}{\lambda^3}\right)^2 - \frac{V}{h^3}\underbrace{\int 4\pi p^2 dp\cdot e^{-2\beta p^2/2m}}_{4\pi\cdot\frac{1}{4}\left(\frac{m}{\beta}\right)^{3/2}\sqrt{\pi} = (\pi m k T)^{3/2} = h^3/\lambda^3\,\frac{1}{2^{3/2}}}$$

$$= \frac{1}{2!}\left(\frac{V}{\lambda^3}\right)^2 - \frac{1}{2^{3/2}}\frac{V}{\lambda^3}$$

2011

SM4 $f(p_x) \propto \left[\#\text{ states with energy } E\text{, x-momentum } p_x\right] \equiv \Omega_{p_x, E}$

$$\Omega_{p_x} = \begin{cases} 0\,, & |p_x| > p = \sqrt{2mE} \\ \frac{V}{h^3} 2\pi \sqrt{p^2 - p_x^2} = \frac{V}{h^3} 2\pi \sqrt{2mE - p_x^2} \end{cases}$$



$\sqrt{p_y^2 + p_z^2} = \sqrt{p^2 - p_x^2}$

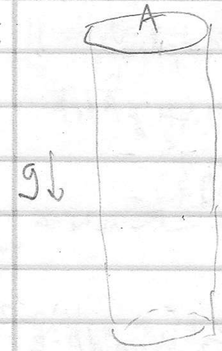
$$f(p_x) = A \begin{cases} \frac{1}{2}\sqrt{2mE - p_x^2}\,, & |p_x| \le \sqrt{2mE} \\ 0\,, & |p_x| \ge \sqrt{2mE} \end{cases}$$

$$1 = \int_{-p}^{p = \sqrt{2mE}} f(p_x)\,dp_x = \frac{A}{2}\int_{-p}^{p} \sqrt{p^2 - p_x^2}\,dp_x \overset{x = p_x/p}{=} \frac{A}{2} p^2 \int_{-1}^{1} \sqrt{1 - x^2}\,dx = \frac{A}{2} p^2 \frac{\pi}{2} \Rightarrow A = \frac{4}{\pi p^2}$$

$$\Rightarrow f(p_x) = \begin{cases} \frac{1}{\pi m E}\sqrt{2mE - p_x^2}\,, & |p_x| \le \sqrt{2mE} \\ 0\,, & |p_x| > \sqrt{2mE} \end{cases}$$

For fixed T, E can vary (and is only exponentially suppressed at large values) and so the distribution is never zero, but still peaks at $p_x = 0$. This is normal for a single-particle distribution. If we were to look at an N-particle distribution, the fixed E and fixed T distributions would approach the same (δ) distribution as $N \to \infty$.

SM5



ideal gases noninteracting $\Rightarrow$ consider independently.

one species with mass m:

$$Q_1 = \sum_{p,z} e^{-\beta\left(\frac{p^2}{2m} + mgz\right)} = \frac{A}{h^3}\int dz \int d^3p\, e^{-\beta\left(\frac{p^2}{2m} + mgz\right)}$$

$$= \frac{A}{V}\int dz\, e^{-\beta mgz} \underbrace{\frac{V}{h^3}\int d^3p\, e^{-\beta\frac{p^2}{2m}}}_{Q_1^{free} = \frac{V}{\lambda^3}} = \frac{A}{\lambda^3}\int dz\, e^{-\beta mgz}$$

$$= \frac{A}{\lambda^3}\frac{kT}{mg}\left[e^{-\beta mg z_{bottom}} - e^{-\beta mg z_{top}}\right]$$

$$Q_N = \frac{1}{N!}\left(\frac{A}{\lambda^3}\frac{kT}{mg}\left[e^{-\beta mg z_{bot}} - e^{-\beta mg z_{top}}\right]\right)^N$$

$$\langle z\rangle = \frac{-1}{Nm\beta}\frac{\partial}{\partial g}\log Q_N = \frac{-kT}{Nm}\frac{\partial}{\partial g}\left[-\log N! + N\log\left(\frac{AkT}{\lambda^3 m}\right) - N\log g + N\log\left(e^{-\beta m g z_{bot}} - e^{-\beta m g z_{top}}\right)\right]$$

$$= \frac{-kT}{Nm}\left(-\frac{N}{g} + N\frac{-\beta m z_{bot} e^{-\beta m g z_{bot}} + \beta m z_{top} e^{-\beta m g z_{top}}}{e^{-\beta m g z_{bot}} - e^{-\beta m g z_{top}}}\right)$$

$$= \frac{kT}{mg} + \frac{z_{bot} e^{-\beta m g z_{bot}} - z_{top} e^{-\beta m g z_{top}}}{e^{-\beta m g z_{bot}} - e^{-\beta m g z_{top}}}, \quad \text{take } z_{bot}=0,\ z_{top}=H$$

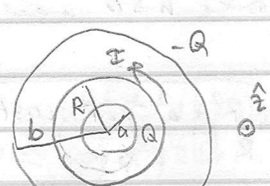
$$= \frac{kT}{mg} - H\frac{e^{-\beta m g H}}{1-e^{-\beta m g H}} = \frac{kT}{mg} - \frac{H}{e^{\frac{mg}{kT}H}-1}$$

n species of mass m_k:

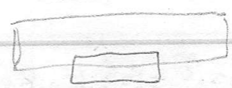
$$z_{CM} = \frac{1}{M}\sum_{k=1}^{n} m_k \langle z\rangle_k = \frac{1}{M}\sum_{k=1}^{n} m_k\left(\frac{kT}{m_k g} - \frac{H}{e^{m_k g/kT\,H}-1}\right), \quad M = \sum_{k=1}^{n} m_k$$

$$= \frac{1}{\sum_{k=1}^{n} m_k}\left[\frac{nkT}{g} - H\sum_{k=1}^{n}\frac{m_k}{e^{m_k g/kT\,H}-1}\right]$$

EMI-1



solenoid



$\ell B = \mu_0 N \ell I \Rightarrow B = \mu_0 N I$ in solenoid

$$2\pi a E = \frac{\partial B}{\partial t}\pi a^2 = \pi a^2 \mu_0 N\frac{dI}{dt}, \quad F = QE, \quad \tau = aF$$

$$\tau_{inner} = aQ\frac{1}{2}a\mu_0 N\frac{dI}{dt} = \frac{\mu_0}{2}a^2 NQ\frac{dI}{dt} \quad (\text{same direction as } I, \text{ opposite } \tfrac{dI}{dt})$$

$$\tau_{outer} = bQ\frac{1}{2}\frac{R^2}{b}\mu_0 N\frac{dI}{dt} = \frac{\mu_0}{2}R^2 NQ\frac{dI}{dt}. \quad (\text{opposite } I, \text{ same dir. as } \tfrac{dI}{dt})$$

$$L = \int \tau\, dt \Rightarrow \quad \vec{L}_{inner} = \frac{\mu_0}{2}a^2 NQI\,\hat{z}$$
$$\vec{L}_{outer} = \frac{\mu_0}{2}R^2 NQI\,(-\hat{z})$$

$$\Rightarrow \vec{L}_{fields} = \frac{\mu_0}{2}(a^2 - R^2)NQI\,\hat{z}$$

2au

EM1-2

λ

$$2\pi\rho l E = \frac{\lambda l}{\epsilon_0} \Rightarrow \vec{E} = \frac{\lambda}{2\pi\epsilon_0}\frac{\hat{\rho}}{\rho} = -\frac{d\Phi}{d\rho}\hat{\rho}$$

$$\Rightarrow \Phi = \frac{-\lambda}{2\pi\epsilon_0}\log\left(\frac{\rho}{\rho_0}\right) \qquad (\rho_0 \text{ arbitrary})$$

$$W\Big|_{a\to b} = \frac{-q\lambda}{2\pi\epsilon_0}\log\left(\frac{b}{a}\right)$$

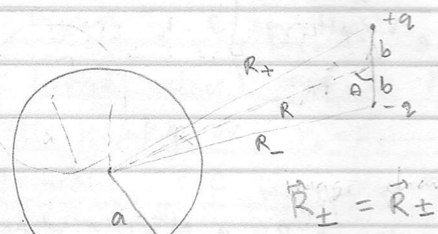
$$b\to\infty \Rightarrow W \to \begin{cases}\infty, & q<0\\ -\infty, & q>0\end{cases}$$

truncated: $$W\Big|_{a\to\infty} = -\int_{-L/2}^{L/2}\frac{1}{4\pi\epsilon_0}\frac{q\lambda\,dz}{\sqrt{a^2+z^2}} = \frac{-q\lambda}{4\pi\epsilon_0}\,a\int_{-L/2a}^{L/2a}\frac{dx}{\sqrt{1+x^2}}, \quad x = z/a, \quad x=\sinh\theta$$

$$= -\frac{q\lambda}{4\pi\epsilon_0}\int\frac{\cosh\theta\,d\theta}{\cosh\theta} = \frac{-q\lambda}{4\pi\epsilon_0}\left(\sinh^{-1}\left(\frac{L}{2a}\right) - \sinh^{-1}\left(-\frac{L}{2a}\right)\right)$$

$$= \frac{-q\lambda}{2\pi\epsilon_0}\sinh^{-1}\left(\frac{L}{2a}\right)$$

EM1-3



model as two charges $\pm q$, $q = \frac{p}{2b}$ and take $b\to 0$ in end.

$$\vec{R}_\pm = \vec{R} \pm b\hat{z}, \quad R_\pm = \sqrt{R^2+b^2\pm 2Rb\cos\theta} \simeq R\sqrt{1\pm 2\tfrac{b}{R}\cos\theta} \simeq R\pm b\cos\theta$$

image charges: $$\frac{q}{|a\hat{r}'-\vec{R}|} = \frac{\frac{a}{R}q}{\left|\frac{a^2}{R}\hat{r}'-a\hat{R}\right|} = \frac{\frac{a}{R}q}{\left|\frac{a^2}{R}\hat{R}-a\hat{r}'\right|}$$

$$\Rightarrow -\frac{a}{R_+}q \text{ at } \frac{a^2}{R_+}\hat{R}_+, \quad \frac{a}{R_-}q \text{ at } \frac{a^2}{R_-}\hat{R}_-$$

$$U = \frac{-\frac{a}{R_+}q^2}{R_+-\frac{a^2}{R_+}} - \frac{\frac{a}{R_-}q^2}{R_- - \frac{a^2}{R_-}} + \frac{\frac{a}{R_+}q^2}{\left|\vec{R}_- - \frac{a^2}{R_+}\hat{R}_+\right|} + \frac{\frac{a}{R_-}q^2}{\left|\vec{R}_+ - \frac{a^2}{R_-}\hat{R}_-\right|}$$

$$= q^2 a\left[\frac{-1}{R_+^2-a^2} - \frac{1}{R_-^2-a^2} + \frac{1}{|\vec{R}_-R_+ - a^2\hat{R}_+|} + \frac{1}{|\vec{R}_+R_- - a^2\hat{R}_-|}\right]$$

$$= \frac{p^2 a}{4b^2}\left[-\frac{1}{R^2-a^2+2R_zb+b^2} - \frac{1}{R^2-a^2-2R_zb+b^2} + \frac{2}{\sqrt{R_-^2R_+^2+a^4-2a^2\vec{R}_+\cdot\vec{R}_-}}\right]$$

$$\vec{R}_+\cdot\vec{R}_- = R^2-b^2, \quad R_-^2R_+^2 = (R^2+b^2+2R_zb)(R^2+b^2-2R_zb)$$
$$= R^4+2R^2b^2+b^4-4R_z^2b^2$$

$$\to 2\left(R^4+2R^2b^2+b^4-4R_z^2b^2+a^4-2a^2R^2+2a^2b^2\right)^{-1/2}$$

$$U = \frac{p^2 a}{4b^2}\frac{1}{R^2-a^2}\left[-\left(1+\frac{2R_z b+b^2}{R^2-a^2}\right)^{-1}-\left(1+\frac{-2R_z b+b^2}{R^2-a^2}\right)^{-1} + 2\left(1+\frac{(2R^2+2a^2-4R_z^2)b^2+b^4}{(R^2-a^2)^2}\right)^{-1/2}\right]$$

$$= \frac{p^2 a}{4b^2}\frac{1}{R^2-a^2}\left[-\left(\cancel{1}-\frac{2R_z b+b^2}{R^2-a^2}+\frac{4R_z^2b^2}{(R^2-a^2)^2}\right)-\left(\cancel{1}-\frac{-2R_z b+b^2}{R^2-a^2}+\frac{4R_z^2b^2}{(R^2-a^2)^2}\right) + \left(\cancel{2}-\frac{(2R^2+2a^2-4R_z^2)b^2}{(R^2-a^2)^2}\right)\right]$$

$$= \frac{p^2 a}{4}\frac{1}{R^2-a^2}\left[\frac{2(R^2-a^2)-8R_z^2-2R^2-2a^2+4R_z^2}{(R^2-a^2)^2}\right]$$

$$= \frac{p^2 a}{(R^2-a^2)^3}\left(-a^2-(\vec{R}\cdot\hat{z})^2\right) = -\frac{p^2 a\left((\vec{R}\cdot\hat{z})^2+a^2\right)}{(R^2-a^2)^3} = -\frac{p^2 a(R^2\cos^2\theta+a^2)}{(R^2-a^2)^3}$$

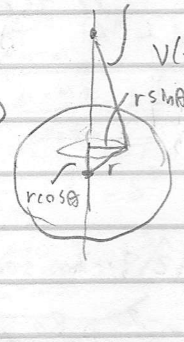
$$\vec{F} = -\nabla U = -\frac{\partial U}{\partial R}\hat{R} - \frac{1}{R}\frac{\partial U}{\partial\theta}\hat{\theta}$$

$$= p^2 a\left(\frac{2R\cos^2\theta(R^2-a^2)^3 - R^2\cos^2\theta\cdot 3(R^2-a^2)^2\cdot 2R}{(R^2-a^2)^6}\hat{R} + \frac{1}{R}\frac{2R^2\cos\theta\sin\theta}{(R^2-a^2)^3}\hat{\theta}\right)$$

$$= \frac{p^2 a}{(R^2-a^2)^4}\left[\left(2R(R^2-a^2)-6R^3\cos^2\theta\right)\hat{R} + 2R(R^2-a^2)\cos\theta\sin\theta\,\hat{\theta}\right]$$

EMI-4 $\rho = \rho_0 \frac{3z^2-r^2}{a^2}$

$= \rho_0 \frac{r^2}{a^2}(3\cos^2\theta - 1)$



$\sqrt{(z-r\cos\theta)^2+r^2\sin^2\theta} = \sqrt{z^2-2zr\cos\theta+r^2}$

$$V(z) = \frac{1}{4\pi\epsilon_0}\int_0^a r^2 dr\int_0^\pi 2\pi\, d\cos\theta\, \rho_0\frac{r^2}{a^2}(3\cos^2\theta-1)\frac{1}{\sqrt{z^2-2zr\cos\theta+r^2}}$$

$$= \frac{\rho_0}{2\epsilon_0 a^2}\int_0^a r^4 dr\int_{-1}^{1} d\mu\,\frac{3\mu^2-1}{\sqrt{r^2+z^2-2zr\mu}}$$

$$= \frac{\rho_0}{2\epsilon_0 a^2}\int_0^a r^4 dr\left[\frac{1}{zr}\sqrt{r^2+z^2-2zr\mu}\,\Big|_{-1}^{1} + 3\int_{-1}^{1}\frac{\mu^2 d\mu}{\sqrt{r^2+z^2-2zr\mu}}\right]$$

Let $x = r^2+z^2-2zr\mu$

$\mu = \frac{r^2+z^2-x}{2zr} = \frac{1}{2}\left(\frac{r}{z}+\frac{z}{r}-\frac{x}{zr}\right) \Rightarrow \mu^2 = \frac{1}{4}\left(\left(\frac{r^2+z^2}{zr}\right)^2+\frac{x^2}{(zr)^2}-2\frac{(r^2+z^2)x}{(zr)^2}\right)$

$d\mu = -\frac{dx}{2zr}$

$$V(z) = \frac{\rho_0}{2\epsilon_0 a^2}\int_0^a r^4 dr\left[\frac{(z-r)-(z+r)}{zr} - \frac{3}{8(zr)^3}\int_{(z+r)^2}^{(z-r)^2}\left[(r^2+z^2)^2x^{-1/2}-2(r^2+z^2)x^{1/2}+x^{3/2}\right]dx\right]$$

$$\left[\frac{-2}{z} - \frac{3}{8(zr)^3}\left\{2(r^2+z^2)^2\underbrace{x^{1/2}}_{-2r} - \frac{4}{3}(r^2+z^2)\underbrace{x^{3/2}}_{(z-r)^3-(z+r)^3} + \frac{2}{5}\underbrace{x^{5/2}}_{(z-r)^5-(z+r)^5}\right\}_{(z+r)^2}^{(z-r)^2}\right]$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

2011

$$V(z) = \frac{\rho_0}{2\epsilon_0 a^2}\int_0^a r^4 dr \left[\frac{-2}{z} - \frac{3}{8(zr)^3}\left\{-4r(r^2+z^2)^2 - \frac{4}{3}(r^2+z^2)\left[(z-r)^3-(z+r)^3\right] + \frac{2}{5}\left[(z-r)^5-(z+r)^5\right]\right\}\right]$$

$$(z\pm r)^5 = z^5 \pm 5z^4 r + 10z^3r^2 \pm 10z^2r^3 + 5zr^4 \pm r^5$$

$$\Rightarrow (z-r)^5 - (z+r)^5 = -10z^4 r - 20z^2r^3 - 2r^5$$

$$(z\pm r)^3 = z^3 \pm 3z^2 r + 3zr^2 \pm r^3 \Rightarrow (z-r)^3 - (z+r)^3 = -6z^2 r - 2r^3$$

$$V(z) = \frac{\rho_0}{2\epsilon_0 a^2}\int_0^a r^4 dr\left[\frac{-2}{z} - \frac{3}{8(zr)^3}\left\{-4r(r^4+z^4+2r^2z^2) - \frac{4}{3}(r^2+z^2)(-6z^2r-2r^3) + \frac{2}{5}(-10z^4r-20z^2r^3-2r^5)\right\}\right]$$

$$\left[-\frac{3}{8z^3}(\cancel{-4z^4}+\cancel{8z^4}\cancel{-4z^4})r^{-2} - \frac{3}{8z}\left(\cancel{\tfrac{16}{3}}\cancel{-8}+\cancel{8}+\cancel{\tfrac{8}{3}}\cancel{-8}\right) - \frac{3}{8}\frac{r^2}{z^3}\left(-4+\frac{8}{3}-\frac{4}{5}\right)\right]$$

$$\frac{-60+40-12}{15} = \frac{-32}{15}$$

$$= \frac{4}{5}\frac{r^2}{z^3}$$

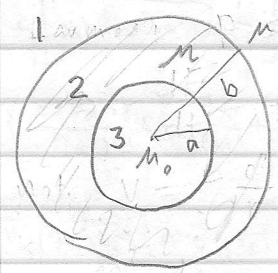
$$V(z) = \frac{\rho_0}{2\epsilon_0 a^2}\frac{4}{5}\frac{1}{z^3}\int_0^a r^6 dr = \frac{2\rho_0}{35\epsilon_0}\frac{a^5}{z^3}$$

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^{-l-1} P_l(\cos\theta)$$

$$V(z) = \sum_{l=0}^{\infty} A_l z^{-l-1} \Rightarrow A_l = \delta_{l2}\frac{2\rho_0}{35\epsilon_0}a^5$$

$$V(r,\theta) = \frac{2\rho_0}{35\epsilon_0}\frac{a^5}{r^3}\frac{3\cos^2\theta - 1}{2}$$

EM1-5



$\vec{B}_0 = B_0\hat{z}$, no source $\Rightarrow \vec{H} = -\nabla\Phi$, $\vec{B} = \mu\vec{H}$

$$\Phi_1 = \left(A r^{-2} - \frac{B_0}{\mu_0}r\right)\cos\theta$$

$$\Phi_2 = (Br + Cr^{-2})\cos\theta$$

$$\Phi_3 = Dr\cos\theta$$

(only $l=1$ terms:)

$\vec{B}_r$ continuous at boundaries $\Rightarrow -\mu H_r = \mu \frac{\partial \Phi}{\partial r}$ cont.

$$-\mu_0 2Ab^{-3} - B_0 = \mu(B - 2Cb^{-3})$$

$$\mu(B - 2Ca^{-3}) = \mu_0 D$$

H_θ cont. $\Rightarrow \frac{\partial \Phi}{\partial \theta}$ cont.

$$Ab^{-2} + \frac{B_0}{\mu_0} b = Bb + Cb^{-2}$$

$$Ba + Ca^{-2} = Da$$

$$\Rightarrow A = b^2(H_0 b + Bb + Cb^{-2}) = Bb^3 + C + H_0 b^3 \quad , H_0 \equiv B_0/\mu_0$$

$$-2\mu_0(B + Cb^{-3} + H_0) - B_0 = \mu(B - 2Cb^{-3})$$

$$\Rightarrow B(-2\mu_0 - \mu) = C(2\mu_0 - 2\mu)b^{-3} + 3B_0 \Rightarrow B = 2\frac{\mu-\mu_0}{2\mu_0+\mu} b^{-3} C - \frac{3}{2\mu_0+\mu} B_0$$

$$B + Ca^{-3} = D$$

$$\Rightarrow \left(2\frac{\mu-\mu_0}{2\mu_0+\mu} b^{-3} + a^{-3}\right) C - \frac{3}{2\mu_0+\mu} B_0 = D$$

$$2\left(\frac{\mu-\mu_0}{2\mu_0+\mu} b^{-3} - a^{-3}\right) C - \frac{3}{2\mu_0+\mu} B_0 = \frac{\mu_0}{\mu} D$$

$$C = \frac{D + \frac{3}{2\mu_0+\mu} B_0}{2\frac{\mu-\mu_0}{2\mu_0+\mu} b^{-3} + a^{-3}}$$

$$\Rightarrow \frac{\mu_0}{\mu} D = 2\frac{\frac{\mu-\mu_0}{2\mu_0+\mu} a^3 - b^3}{2\frac{\mu-\mu_0}{2\mu_0+\mu} a^3 + b^3}\left(D + \frac{3}{2\mu_0+\mu} B_0\right) - \frac{3}{2\mu_0+\mu} B_0$$

$$\underbrace{\left(\frac{\mu_0}{\mu} - \frac{\frac{\mu-\mu_0}{2\mu_0+\mu} a^3 - b^3}{\frac{\mu-\mu_0}{2\mu_0+\mu} a^3 + b^3/2}\right)}_{\frac{\left(\frac{\mu_0}{\mu}-1\right)\frac{\mu-\mu_0}{2\mu_0+\mu} a^3 + \left(\frac{\mu_0}{2\mu}+1\right) b^3}{\frac{\mu-\mu_0}{2\mu_0+\mu} a^3 + b^3/2}} D = \frac{-3}{2\mu_0+\mu} B_0 \underbrace{\left(1 - \frac{\frac{\mu-\mu_0}{2\mu_0+\mu} a^3 - b^3}{\frac{\mu-\mu_0}{2\mu_0+\mu} a^3 + \frac{b^3}{2}}\right)}_{\frac{\frac{3}{2} b^3}{\frac{\mu-\mu_0}{2\mu_0+\mu} a^3 + b^3/2}}$$

$$D = -\frac{3\mu}{2\mu_0+\mu} B_0 \frac{3/2\, b^3}{(\mu_0-\mu)\frac{\mu-\mu_0}{2\mu_0+\mu} a^3 + \left(\frac{\mu_0}{2}+\mu\right) b^3} = -\frac{3\mu}{2\mu_0+\mu} \frac{3 B_0}{\mu_0 + 2\mu - \frac{2(\mu-\mu_0)^2}{2\mu_0+\mu}\left(\frac{a}{b}\right)^3}$$

$$\Phi_3 = Dz \Rightarrow \vec{B}_3 = -D\hat{z} = \frac{9\mu \vec{B}_0}{(2\mu_0+\mu)(\mu_0+2\mu) - 2(\mu-\mu_0)^2 (a/b)^3}$$

2011

$$\vec{B}_3 = \frac{9\mu\vec{B}_0}{2(\mu_0^2+\mu^2)(1-(\frac{a}{b})^3) + \mu\mu_0(5+4(\frac{a}{b})^2)}$$

$$\mu \gg \mu_0 \Rightarrow \vec{B}_3 \simeq \frac{9\vec{B}_0}{2\mu(1-(\frac{a}{b})^3)} = \frac{9}{2(1-(\frac{a}{b})^3)}\frac{\vec{B}_0}{\mu}$$

QM1-1 $|+x\rangle = \frac{1}{\sqrt{2}}(|+z\rangle + |-z\rangle)$

$$H = -\vec{\mu}\cdot\vec{B} = -\mu B\sigma_z, \quad U = e^{-\frac{i}{\hbar}HT} = e^{\frac{i\mu B}{\hbar}T\sigma_z}$$

$$U|+x\rangle = \frac{1}{\sqrt{2}}\left(e^{\frac{i\mu BT}{\hbar}}|+z\rangle + e^{-\frac{i\mu BT}{\hbar}}|-z\rangle\right)$$

$$H_2 = -\mu B\sigma_y, \quad |\pm y\rangle = \frac{1}{\sqrt{2}}(|+z\rangle \pm i|-z\rangle)$$

$$\Rightarrow |+z\rangle = \frac{1}{\sqrt{2}}(|+y\rangle + |-y\rangle), \quad |-z\rangle = \frac{-i}{\sqrt{2}}(|+y\rangle - |-y\rangle)$$

$$U|+x\rangle = \frac{1}{2}\left((e^{i\mu BT/\hbar} - ie^{-i\mu BT/\hbar})|+y\rangle + (e^{i\mu BT/\hbar} + ie^{-i\mu BT/\hbar})|-y\rangle\right)$$

$$U_2 = e^{i\mu BT/\hbar\,\sigma_z}$$

$$U_2U|+x\rangle = \frac{1}{2}\left((e^{2i\mu BT/\hbar} - i)|+y\rangle + (1 + ie^{-2i\mu BT/\hbar})|-y\rangle\right)$$

note $\langle +x|\pm y\rangle = \frac{1}{2}(\langle +z| + \langle -z|)(|+z\rangle \pm i|-z\rangle) = \frac{1}{2} \pm \frac{i}{2}$

$$\langle +x|U_2U|+x\rangle = \frac{1}{2}\left[(e^{2i\mu BT/\hbar} - i)(\tfrac{1}{2}+\tfrac{i}{2}) + (1+ie^{-2i\mu BT/\hbar})(\tfrac{1}{2}-\tfrac{i}{2})\right]$$

$$= \frac{1}{4}\left[e^{2i\mu BT/\hbar}(1+i) + e^{-2i\mu BT/\hbar}(1+i) - i + 1 + 1 - i\right]$$

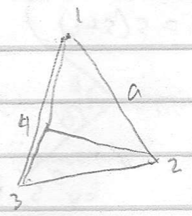
$$= \frac{1}{2}\left[(1+i)\cos(2\mu BT/\hbar) + 1 - i\right]$$

$$|\langle +x|U_2U|+x\rangle|^2 = \frac{1}{4}\left[(1+\cos(2\mu BT/\hbar))^2 + (1-\cos(2\mu BT/\hbar))^2\right]$$

$$= \frac{1}{2}\left[\cos^2(2\mu BT/\hbar) + 1\right]$$

$$\begin{Bmatrix} 6\hbar^2 \\ 2\hbar^2 \\ 0 \end{Bmatrix} = S^2 = (S_1+S_2+S_3+S_4)^2 = \sum_i S_i^2 + 2\sum_{i<j} \bar{S}_i\cdot\bar{S}_j = 3\hbar^2 + 2\sum_{i<j}\bar{S}_i\cdot\bar{S}_j$$

QM1-2



$$H_{ij} = A\,\frac{\bar{S}_i\cdot\bar{S}_j}{r_{ij}^3} = \frac{A}{a^3}\,\bar{S}_i\cdot\bar{S}_j \Rightarrow \langle H\rangle = \frac{A}{a^3}\,\frac{1}{2}\left(\begin{Bmatrix} 6\hbar^2 \\ 2\hbar^2 \\ 0 \end{Bmatrix} - 3\hbar^2\right),$$

$$\langle H\rangle = \frac{A}{a^3}\,\frac{1}{2}\begin{Bmatrix} 3\hbar^2 \\ -\hbar^2 \\ -3\hbar^2 \end{Bmatrix}$$

QM1-3 $a|z\rangle = z|z\rangle$, $a = \frac{1}{\sqrt{2m}}(p + im\omega x)$

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}\ , \quad a + a^\dagger = \frac{1}{\sqrt{2m}}(2p) = \sqrt{\frac{2}{m}}\,p \Rightarrow p = \sqrt{\frac{m}{2}}(a + a^\dagger)$$

$$a - a^\dagger = \frac{im\omega}{\sqrt{2m}}(2x) = i\sqrt{2m}\,\omega x \Rightarrow x = \frac{-i}{\sqrt{2m}\,\omega}(a - a^\dagger)$$

$$\langle x\rangle = \frac{-i}{\sqrt{2m}\,\omega}\Big(\underbrace{\langle z|a|z\rangle}_{z|z\rangle} - \underbrace{\langle z|a^\dagger|z\rangle}_{\langle z|z^*}\Big) = \frac{-i}{\sqrt{2m}\,\omega}(z - z^*)$$

$$\langle p\rangle = \sqrt{\frac{m}{2}}(z + z^*)$$

$$\langle x^2\rangle = \frac{-1}{2m\omega^2}\Big(\underbrace{\langle z|a^2|z\rangle}_{z^2} + \underbrace{\langle z|(a^\dagger)^2|z\rangle}_{(z^*)^2} - 2\underbrace{\langle z|a^\dagger a|z\rangle}_{|z|^2} - \underbrace{\langle z|[a,a^\dagger]|z\rangle}_{-\hbar\omega}\Big)$$

$$[a, a^\dagger] = \frac{1}{2m}\underbrace{[p + im\omega x,\ p - im\omega x]}_{im\omega(i\hbar) - im\omega(-i\hbar) = -2m\omega\hbar} = -\hbar\omega$$

$$= \frac{-1}{2m\omega^2}\left(z^2 + (z^*)^2 - 2zz^* + \hbar\omega\right) = \frac{-1}{2m\omega^2}\left((z - z^*)^2 + \hbar\omega\right)$$

$$\langle p^2\rangle = \frac{m}{2}\left(\langle a^2\rangle + \langle a^\dagger\rangle^2 + 2\langle a^\dagger a\rangle - \langle [a,a^\dagger]\rangle\right)$$

$$= \frac{m}{2}\left(z^2 + (z^*)^2 + 2|z|^2 + \hbar\omega\right) = \frac{m}{2}\left((z + z^*)^2 + \hbar\omega\right)$$

$$\sigma_x = \frac{1}{\sqrt{2m}\,\omega}\left(-(z - z^*)^2 + \hbar\omega + (z - z^*)^2\right) = \frac{\sqrt{\hbar\omega}}{\sqrt{2m}\,\omega} = \sqrt{\frac{\hbar}{2m\omega}}$$

$$\sigma_p = \sqrt{\frac{m\hbar\omega}{2}} \qquad \Rightarrow \quad \sigma_p = m\omega\,\sigma_x$$

$$\sigma_x\sigma_p = \frac{\hbar}{2}$$

2011 QMI-4 $s = \frac{3}{2}$ (mult. 4) or $\frac{1}{2}$ (mult. 4) $\left(\langle S^2 \rangle = s(s+1) \right)$

$|\uparrow\uparrow\uparrow\rangle$: $s = \frac{3}{2}$, $m = \frac{3}{2}$

$\frac{1}{\sqrt{3}}(|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle)$: $s = \frac{3}{2}$, $m = \frac{1}{2}$ by $S_-|\uparrow\uparrow\uparrow\rangle$

$\frac{1}{\sqrt{3}}(|\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle)$: $s = \frac{3}{2}$, $m = -\frac{1}{2}$ by $S_+|\downarrow\downarrow\downarrow\rangle$

$|\downarrow\downarrow\downarrow\rangle$: $s = \frac{3}{2}$, $m = -\frac{3}{2}$

$$\left.\begin{array}{l}\frac{1}{\sqrt{2}}(|\uparrow\downarrow\uparrow\rangle - |\uparrow\uparrow\downarrow\rangle) \\ \frac{1}{\sqrt{6}}(|\uparrow\downarrow\uparrow\rangle + |\uparrow\uparrow\downarrow\rangle - 2|\downarrow\uparrow\uparrow\rangle)\end{array}\right\} s = \frac{1}{2},\ m = \frac{1}{2}$$

(orthogonal to $s = \frac{3}{2}$ states)

$$\left.\begin{array}{l}\frac{1}{\sqrt{2}}(|\downarrow\uparrow\downarrow\rangle - |\downarrow\downarrow\uparrow\rangle) \\ \frac{1}{\sqrt{6}}(|\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle - 2|\uparrow\downarrow\downarrow\rangle)\end{array}\right\} s = \frac{1}{2},\ m = -\frac{1}{2}$$

QMI-5 $V = \begin{cases} \frac{1}{2}m\omega^2 x^2 & ; x > 0 \\ +\infty & , x < 0 \end{cases}$: harmonic oscillator with only odd wavefunctions allowed.

but n odd $\Leftrightarrow$ $\psi_n(x)$ odd

$\Rightarrow$ $E = (n + \frac{1}{2})\hbar\omega$ with $n = 1, 3, 5, 7, 9, \ldots$

EMII-1 $(\bar{E} \times \bar{B})^2 = \underbrace{\epsilon_{ijk}\epsilon_{lmk}}_{\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}} E_i E_l B_j B_m = E_i E_i B_j B_j - E_i E_j B_i B_j = \bar{E}^2 \bar{B}^2 - (\bar{E}\cdot\bar{B})^2$

$(|E|-|B|)^2 = E^2 + B^2 - 2EB$

$$= \frac{1}{4}\Big[E^2 + B^2 - \underbrace{(|E|-|B|)^2}_{\le 0}\Big]^2 - \underbrace{(\bar{E}\cdot\bar{B})^2}_{\le 0} \le \frac{1}{4}(E^2 + B^2)^2$$

with equality when $E^2 = B^2$ and $\bar{E}\cdot\bar{B} = 0$

but $|\vec{P}|c = \mathcal{E}$ $\Rightarrow$ $|\bar{E}\times\bar{B}| = \frac{1}{2}(E^2 + B^2)$ $\Rightarrow$ $(\bar{E}\times\bar{B})^2 = \frac{1}{4}(E^2 + B^2)$

so $E^2 = B^2$ and $\bar{E}\cdot\bar{B} = 0$

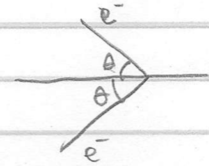
EMII-2 $\omega = \frac{2\pi c}{\lambda}$, $\vec{E} = \vec{E}_0 e^{-i\omega t}$ $\Rightarrow$ $\vec{d} = \frac{\epsilon - 1}{\epsilon + 2} a^3 \vec{E}_0 e^{-i\omega t} \equiv \vec{d}_0 e^{-i\omega t}$

treat as charge $q = \frac{d_0}{b}$ at position $x = be^{-i\omega t}$

$$P = \frac{2}{3}\frac{q^2}{c^3}|\ddot{x}|^2 = \frac{2}{3}\frac{d_0^2}{\cancel{b^2} c^3}\omega^4 \cancel{b^2} = \frac{2}{3}\frac{(2\pi)^4 c}{\lambda^4}\left(\frac{\epsilon - 1}{\epsilon + 2}\right)^2 a^6 |\vec{E}_0|^2$$

but incident $S = \frac{c}{4\pi}|\vec{E}_0|^2$ $\Rightarrow$ $\sigma = \frac{P}{S} = \frac{4}{3}(2\pi)^5\left(\frac{\epsilon - 1}{\epsilon + 2}\right)^2 \frac{a^6}{\lambda^4}$

EM11-3



rest frame: $E_e = m_e + m_\pi = \gamma m_e \Rightarrow \gamma = 1 + \frac{m_\pi}{m_e}$

transform to frame where $= (1-u'^2)^{-1/2}$

$\frac{dy}{dx} = \pm \tan\theta$

$$\left.\begin{aligned} dt &= \gamma dt' - \gamma v \underbrace{dx'}_{0} \\ dx &= -\gamma v dt' + \gamma \underbrace{dx'}_{0} \\ dy &= dy' \end{aligned}\right\}$$

$$\frac{dy}{dx} = \frac{dy'}{-\gamma_v v dt'} = \pm \frac{u'}{\gamma_v v} = \pm \tan\theta \Rightarrow \gamma_v v = \frac{u'}{\tan\theta}$$

where $u' = \left(1 - (1 + \frac{m_\pi}{m_e})^{-2}\right)^{1/2}$

$$v\gamma_v = \frac{v}{\sqrt{1-v^2}} = \frac{1}{\sqrt{\frac{1}{v^2} - 1}} = (v^{-2} - 1)^{-1/2}$$

$$\Rightarrow v = \left((v\gamma_v)^{-2} + 1\right)^{-1/2}$$

$$= \left(\frac{\tan^2\theta}{1 - (1 + \frac{m_\pi}{m_e})^{-2}} + 1\right)^{-1/2}$$

$$u^2 = \left(\frac{dy}{dt}\right)^2 + v^2$$

$$\frac{dy}{dt} = \frac{dy'}{\gamma_v dt'} = \frac{u'}{\gamma_v}$$

$$\tan\theta = \frac{u'/\gamma_v}{v} = \frac{u'}{\gamma_v v}$$

$$\gamma' = \gamma_v = (1-v^2)^{-1/2} = \left(1 - \frac{1}{1 + \tan^2\theta / 1 - (1 + m_\pi/m_e)^{-2}}\right)^{-1/2}$$

$$\gamma = \left(1 - \frac{u'^2}{\gamma'^2} - v^2\right)^{-1/2}$$

$$\gamma' = \left(\frac{\tan^2\theta / 1 - (1 + m_\pi/m_e)^{-2}}{1 + \tan^2\theta / 1 - (1 + m_\pi/m_e)^{-2}}\right)^{-1/2}$$

$$= \frac{\left(1 - (1 + m_\pi/m_e)^{-2} + \tan^2\theta\right)^{1/2}}{\tan\theta}$$

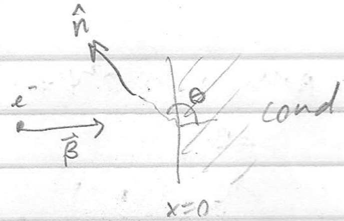
$$\gamma = \left(1 - \frac{(1 - (1 + m_\pi/m_e)^{-2})\tan^2\theta}{1 - (1 + m_\pi/m_e)^{-2} + \tan^2\theta} - \frac{1 - (1 + m_\pi/m_e)^{-2}}{1 - (1 + m_\pi/m_e)^{-2} + \tan^2\theta}\right)^{-1/2}$$

$$= \left(\frac{\tan^2\theta - (1 - (1 + m_\pi/m_e)^{-2})\tan^2\theta}{1 - (1 + m_\pi/m_e)^{-2} + \tan^2\theta}\right)^{-1/2} = \frac{1 + m_\pi/m_e}{\tan\theta} \cdot \sqrt{1 - (1 + \frac{m_\pi}{m_e})^{-2} + \tan^2\theta}$$

$\theta = 0$: $\gamma \to \infty$ (since $\tan\theta \to 0$). $\gamma \sim \frac{1}{\theta}\sqrt{(1 + \frac{m_\pi}{m_e})^2 - 1}$

$\theta = \pi/2$: $\gamma = 1 + \frac{m_\pi}{m_e}$ (since $\tan\theta \to \infty$)

2011 EMII-4



$\hat{n}\times\vec{\beta} = -\beta\sin\theta\,\hat{z} \Rightarrow \hat{n}\times(\hat{n}\times\vec{\beta}) = -\beta\sin^2\theta\,\hat{x} + \beta\sin\theta\cos\theta\,\hat{y}$

$$\frac{dW}{d\omega\, d\Omega} = \frac{e^2\omega^2}{4\pi^2 c}\left|\underbrace{\int_{-\infty}^{0} dt'\,\beta\sin\theta(-\sin\theta\,\hat{x}+\cos\theta\,\hat{y})\,e^{i\omega t'(1-\beta\cos\theta)-\epsilon|t'|}}_{\frac{\beta\sin\theta(-\sin\theta\,\hat{x}+\cos\theta\,\hat{y})}{i\omega(1-\beta\cos\theta)}}\right|^2$$

$= \dfrac{e^2}{4\pi^2 c}\dfrac{\beta^2\sin^2\theta}{(1-\beta\cos\theta)^2}$ which peaks sharply in the forward direction ($\sim\theta^{-2}$)

But due to perfect conductor,
there is an image electron (with positive charge) moving in the opposite direction with $\theta\to\pi-\theta$, $\sin\theta\to\sin\theta$, $\cos\theta\to-\cos\theta$

$$\Rightarrow \frac{dW}{d\omega\, d\Omega} = \frac{e^2}{4\pi^2 c}\frac{\beta^2\sin^2\theta}{(1+\beta\cos\theta)^2}$$

EMII-5 $\nabla\times\vec{B} = -i\omega\vec{E} \Rightarrow \vec{e}_3\cdot\nabla_t\times\vec{B} = -i\omega a e^{ikz-i\omega t}$

$\vec{e}_3\cdot\nabla_t\times\vec{B} = \dfrac{\partial B_y}{\partial x} - \dfrac{\partial B_x}{\partial y}$, $i\omega\vec{B} = \nabla\times\vec{E}$

$$\vec{B} = \frac{1}{i\omega}\left(\frac{1}{\sqrt{2}}(\overset{\text{small}}{\nabla E_0} + ikE_0\vec{e}_3)e^{ikz-i\omega t}\times(\vec{e}_1+i\vec{e}_2) + (\nabla a + ik\vec{e}_3 a)e^{ikz-i\omega t}\times\vec{e}_3\right)$$

$$= \frac{1}{i\omega}\left(\frac{\vec{e}_1+i\vec{e}_2}{\sqrt{2}}kE_0 e^{ikz-i\omega t} + \nabla a\times\vec{e}_3 e^{ikz-i\omega t}\right), \quad \nabla a\times\vec{e}_3 = -\frac{\partial a}{\partial x}\vec{e}_2 + \frac{\partial a}{\partial y}\vec{e}_1$$

$$\vec{e}_3\cdot\nabla_t\times\vec{B} = \frac{1}{i\omega}\left(\frac{1}{\sqrt{2}}\left(i\frac{\partial E_0}{\partial x} - \frac{\partial E_0}{\partial y}\right)k e^{ikz-i\omega t} - \left(\frac{\partial^2 a}{\partial x^2} + \frac{\partial^2 a}{\partial y^2}\right)e^{ikz-i\omega t}\right)$$

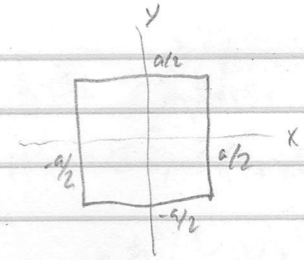
$$\Rightarrow \omega^2 a = \frac{k}{\sqrt{2}}\left(i\frac{\partial E_0}{\partial x} - \frac{\partial E_0}{\partial y}\right) - \underbrace{\left(\frac{\partial^2 a}{\partial x^2} + \frac{\partial^2 a}{\partial y^2}\right)}_{\text{also small}}$$

$$a \simeq \frac{k}{\sqrt{2}\omega^2}\left(i\frac{\partial E_0}{\partial x} - \frac{\partial E_0}{\partial y}\right)$$

If $a=0$ then $\vec{E}_0$ is constant, contradicting that it drops off.

Approximate because it will spread out as z increases (E_0, a depend on z)

QMII-1



$-\frac{\hbar^2}{2m}\nabla^2\psi = E\psi$, i.e. $\nabla^2\psi = -k^2\psi$, $k = \sqrt{\frac{2mE}{\hbar^2}}$

$(\partial_x^2 + \partial_y^2 + k^2)\psi = 0$

$\Rightarrow \psi = C\cos\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{a}\right)$, $k^2 = \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{a}\right)^2$

$C = \frac{2}{a}$ $\qquad E_{nm} = \frac{(n^2+m^2)\pi^2\hbar^2}{2ma^2}$

ground: $\psi_0 = \frac{2}{a}\cos\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi y}{a}\right)$

1st excited: $\psi_1^{(1)} = \frac{2}{a}\cos\left(\frac{2\pi x}{a}\right)\cos\left(\frac{\pi y}{a}\right)$, $\psi_1^{(2)} = \frac{2}{a}\cos\left(\frac{\pi x}{a}\right)\cos\left(\frac{2\pi y}{a}\right)$

$$\langle V'\rangle_0 = \frac{4}{a^2}A\int_{-a/2}^{a/2}dx\int_{-a/2}^{a/2}dy\,\cos^2\left(\frac{\pi x}{a}\right)\cos^2\left(\frac{\pi y}{a}\right)e^{-x^2/b^2}$$

$$\simeq \frac{4}{a^2}A\,\frac{a}{2}\int_{-\infty}^{\infty}e^{-x^2/b^2}dx \quad \text{since } b \ll a \quad \left(\cos^2\left(\frac{\pi b}{a}\right)\simeq 1\right)$$

$$\simeq \frac{2}{a}Ab\sqrt{\pi}$$, same for $\langle V'\rangle_1^{(1,2)}$ (under approximation)

so $E_0 = \frac{\pi^2\hbar^2}{ma^2} + \frac{2\sqrt{\pi}Ab}{a}$, deg. 1

$E_1 = \frac{5}{2}\frac{\pi^2\hbar^2}{ma^2} + \frac{2\sqrt{\pi}Ab}{a}$, deg. 2

QMII-2 $i\frac{d}{dt}U(t;t_0)|\psi(t_0)\rangle_I = H_I(t)U(t;t_0)|\psi(t_0)\rangle_I$

$\Rightarrow i\frac{dU(t;t_0)}{dt} = H_I(t)U(t;t_0)$, since $|\psi(t_0)\rangle_I$ can be any state

$\Rightarrow i\frac{U(t+dt)-U(t)}{dt} = H_I(t)U(t) \Rightarrow U(t+dt) = U(t)(1-i\,dt\,H_I(t))$

so $U(t) = \prod_{\substack{t'\in(t_0,t) \\ \text{with interval} \\ dt'\to 0}}(1-iH_I(t')dt')$, then carry out binomial multiplication

$$U(t;t_0) = 1 - i\underbrace{\int_{t_0}^{t}H_I(t')dt'}_{\sum_{t'}} - \underbrace{\int_{t_0}^{t}dt'\int_{t_0}^{t'}dt''}_{\sum_{t''<t'}}H_I(t')H_I(t'') + i\underbrace{\int_{t_0}^{t}dt'\int_{t_0}^{t'}dt''\int_{t_0}^{t''}dt'''}_{\sum_{t'''<t''<t'}}H_I(t')H_I(t'')H_I(t''') + \dots$$

note this satisfies $i\frac{dU(t)}{dt} = H_I(t)U(t)$

2011 QM11-3 $V' = -Ex$, $0 < t < \pi/\omega$, $n=0$ initially

$\langle n|V'|0\rangle = -E\langle n|x|0\rangle$, $x = \sqrt{\frac{\hbar}{2m\omega}}(a+a^\dagger)$

since $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = (a^\dagger a + \frac{1}{2})\hbar\omega$

$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right)$

$V'_{n0} = -E\sqrt{\frac{\hbar}{2m\omega}}\left(\cancel{\langle n|a|0\rangle} + \underbrace{\langle n|a^\dagger|0\rangle}_{\delta_{n1}}\right) = -E\sqrt{\frac{\hbar}{2m\omega}}\,\delta_{n1}$

$c_n = -E\sqrt{\frac{\hbar}{2m\omega}}\,\delta_{n1}\left(\frac{-i}{\hbar}\right)\underbrace{\int_0^{\pi/\omega} dt' e^{i\omega t'}}_{\frac{1}{i\omega}(e^{i\pi}-1) = \frac{-2}{i\omega}} = +E\sqrt{\frac{\hbar}{2m\omega}}\,\delta_{n1}\frac{2}{\omega}$

$|c_n|^2 = \delta_{n1}\frac{E}{\omega}\sqrt{\frac{2\hbar}{m\omega}}$

QM11-4 $\pi\vec{x} = -\vec{x}\pi$, $\pi\vec{p} = -\vec{p}\pi$ $\Rightarrow$ $\pi\vec{L} = \pi(\vec{x}\times\vec{p}) = (-\vec{x})\times(-\vec{p})\pi = \vec{x}\times\vec{p}\,\pi$
$= \vec{L}\,\pi$

SHO: $a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$, $a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{ip}{m\omega}\right)$ $\Rightarrow$ $\pi a^\dagger = -a^\dagger\pi$

$\Rightarrow \pi|n+1\rangle = \frac{1}{\sqrt{n+1}}\pi a^\dagger|n\rangle = \frac{-1}{\sqrt{n+1}}a^\dagger\pi|n\rangle$

Suppose $\pi|n\rangle = (-1)^n|n\rangle$. Then $\pi|n+1\rangle = \frac{(-1)^{n+1}}{\sqrt{n+1}}a^\dagger|n\rangle$
$= (-1)^{n+1}|n+1\rangle$.

Since $\pi|0\rangle = |0\rangle = (-1)^0|0\rangle$
then by induction $\pi|n\rangle = (-1)^n|n\rangle$ $\forall n\in\mathbb{N}$

$\theta\vec{x} = \vec{x}\theta$, $\theta\vec{p} = -\vec{p}\theta$ $\Rightarrow$ $\theta\vec{L} = -\vec{L}\theta$

$\theta|\alpha\rangle = a^*\underbrace{\theta|+\rangle}_{|-\rangle} + b^*\underbrace{\theta|-\rangle}_{-|+\rangle} = -b^*|+\rangle + a^*|-\rangle$

$\theta|\alpha\rangle = |\alpha\rangle \Rightarrow a^* = b$, $b^* = -a$ $\Rightarrow a = -a \Rightarrow a = 0 \Rightarrow b = 0$
$\Rightarrow$ cannot be invariant

QM11-5 $\vec{S}_1\cdot\vec{S}_2 = \frac{\hbar^2}{4}\left(\begin{pmatrix} &&&1\\ &&1&\\ &1&&\\ 1&&& \end{pmatrix} + \begin{pmatrix} &&&-1\\ &&1&\\ &1&&\\ -1&&& \end{pmatrix} + \begin{pmatrix} 1&&&\\ &-1&&\\ &&-1&\\ &&&1 \end{pmatrix}\right) = \frac{\hbar^2}{4}\begin{pmatrix} 1&&&\\ &-1&2&\\ &2&-1&\\ &&&1 \end{pmatrix}$

$H' = J\vec{S}_1\cdot\vec{S}_2 - B(\alpha S_{1z} + \beta S_{2z})$ (no orbital contribution since $l=0$)

$$= J\frac{\hbar^2}{4}\begin{pmatrix} 1&&&\\ &-1&2&\\ &2&-1&\\ &&&1 \end{pmatrix} - B\frac{\hbar}{2}\begin{pmatrix} \alpha+\beta&&&\\ &\alpha-\beta&&\\ &&-\alpha+\beta&\\ &&&-\alpha-\beta \end{pmatrix}$$

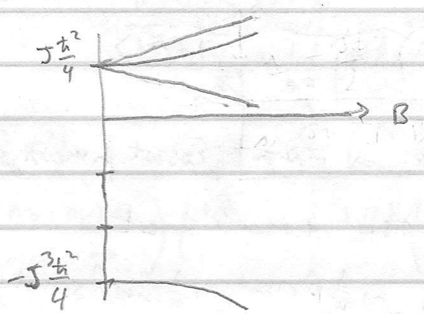
$$= \frac{\hbar}{2}\begin{pmatrix} \frac{J\hbar}{2} - B\alpha - B\beta &&&\\ & -\frac{J\hbar}{2} - B\alpha + B\beta & J\hbar &\\ & J\hbar & -\frac{J\hbar}{2} + B\alpha - B\beta &\\ &&& \frac{J\hbar}{2} + B\alpha + B\beta \end{pmatrix}$$

eigenvalues $\frac{\hbar}{2}\lambda$, $0 = \left(\frac{J\hbar}{2} - B\alpha - B\beta - \lambda\right)\left(\frac{J\hbar}{2} + B\alpha + B\beta - \lambda\right)\left[\left(\frac{J\hbar}{2} + B\alpha - B\beta + \lambda\right)\left(\frac{J\hbar}{2} - B\alpha + B\beta + \lambda\right) - (J\hbar)^2\right]$

$$0 = \left(\left(\frac{J\hbar}{2} - \lambda\right)^2 - (B\alpha + B\beta)^2\right)\left(\left(\frac{J\hbar}{2} + \lambda\right)^2 - (B\alpha - B\beta)^2 - (J\hbar)^2\right)$$

$$\Rightarrow \lambda = \frac{J\hbar}{2} \pm B(\alpha+\beta), \quad \lambda = -\frac{J\hbar}{2} \pm \sqrt{B^2(\alpha-\beta)^2 + J^2\hbar^2}$$

with $E = \frac{\hbar}{2}\lambda$ $\qquad \simeq -\frac{J\hbar}{2} \pm J\hbar\left(1 + \frac{B^2(\alpha-\beta)^2}{2J^2\hbar^2}\right)$



$E = \frac{\hbar}{2}\left(\frac{J\hbar}{2} \pm B(\alpha+\beta)\right)$ corresponds to $\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = |\downarrow\downarrow\rangle$ and $\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = |\uparrow\uparrow\rangle$ respectively

$E = \frac{\hbar}{2}\left(-\frac{J\hbar}{2} \pm \sqrt{B^2(\alpha-\beta)^2 + J^2\hbar^2}\right)$; $\begin{pmatrix} -B(\alpha-\beta) \mp \sqrt{\ldots} & J\hbar \\ J\hbar & B(\alpha-\beta) \mp \sqrt{\ldots} \end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = 0$

$\left(-B(\alpha-\beta) \mp \sqrt{\ldots}\right)a + J\hbar b = 0 \qquad \Rightarrow a = \frac{J\hbar}{B(\alpha-\beta) \pm \sqrt{\ldots}}\, b$

$J\hbar a + \left(B(\alpha-\beta) \mp \sqrt{\ldots}\right)b = 0$

i.e. $\begin{pmatrix}0\\a\\b\\0\end{pmatrix} = a|\uparrow\downarrow\rangle + b|\downarrow\uparrow\rangle$ where $b = \frac{B(\alpha-\beta) \pm \sqrt{B^2(\alpha-\beta)^2 + J^2\hbar^2}}{J\hbar}\, a$