2010

**QM1-1** consider simultaneous eigenstate $|\psi\rangle$ of $A$ and $C$:

$$A|\psi\rangle = a|\psi\rangle, \quad C|\psi\rangle = c|\psi\rangle$$

Now $CB|\psi\rangle = BC|\psi\rangle = Bc|\psi\rangle = cB|\psi\rangle$

so $|\psi\rangle$ and $B|\psi\rangle$ are both eigenstates of $C$ with eigenvalue $c$. If they are the same state, then $|\psi\rangle$ is also an eigenstate of $B$. Suppose this is true for all simultaneous eigenstates $|\psi\rangle$ of $A$ and $C$. Then these form a complete set of simultaneous eigenstates of $A$ and $B$, contradicting $[A,B]\neq 0$. Thus there must be a $|\psi\rangle$ for which $|\psi\rangle$ and $B|\psi\rangle$ are different states, so they are degenerate in $C$.

**QM1-2** $|\psi\rangle$ is translated by $\lambda$ from $|\phi\rangle$

$$|\psi(t)\rangle = \underbrace{U(t)\, e^{-ip\lambda/\hbar}\, U^\dagger(t)}_{e^{-ip\lambda/\hbar}\big|_{-t} = e^{-ip(-t)\lambda/\hbar}}\, \underbrace{U(t)|\phi\rangle}_{|\phi(t)\rangle} = e^{-ip(-t)\lambda/\hbar}\, |\phi(t)\rangle$$

note $\frac{dp}{dt} = -m\omega^2 x \Rightarrow p = p_0\cos\omega t - m\omega x_0 \sin\omega t$

so $\hat{p}(-t) = \hat{p}\cos\omega t + m\omega\hat{x}\sin\omega t$

$$\langle x|\psi(t)\rangle = \langle x|\underbrace{e^{-i\frac{\lambda}{\hbar}\hat{p}\cos\omega t - i\frac{\lambda}{\hbar}m\omega\hat{x}\sin\omega t}}_{e^{-i\frac{\lambda}{\hbar}\hat{p}\cos\omega t}\, e^{-i\frac{\lambda}{\hbar}m\omega\hat{x}\sin\omega t}\, e^{+\frac{1}{2}\frac{\lambda^2}{\hbar^2}m\omega\cos\omega t\sin\omega t(-i\hbar)}}|\phi(t)\rangle$$

$$= \langle x-\lambda\cos\omega t|\, e^{-i\frac{\lambda}{\hbar}m\omega\hat{x}\sin\omega t}\, e^{-\frac{i}{2\hbar}\lambda^2 m\omega\cos\omega t\sin\omega t}\, |\phi(t)\rangle$$

$$= \underbrace{e^{-i\frac{\lambda}{\hbar}m\omega x\sin\omega t}\, e^{\frac{i}{2\hbar}\lambda^2 m\omega\cos\omega t\sin\omega t}}_{}\, \langle x-\lambda\cos\omega t|\phi(t)\rangle$$

no $x^2$ term in exponent $\Rightarrow$ does not spread

QM1-3 $\psi(0)=0, \quad \frac{d^2\psi}{dx^2} = k^2\psi$ where $k = \sqrt{\frac{-2mE}{\hbar^2}}$ for $x \in (0,a)$

$\Rightarrow \psi(x) \propto \sinh(kx)$, say $\psi(x) = \begin{cases} A\sinh(kx), & x<a \\ Be^{-k(x-a)} & x>a \end{cases}$

$-\frac{\hbar^2}{2m}\psi'' - \alpha\delta(x-a)\psi = E\psi, \quad \psi'' + \frac{2m\alpha}{\hbar^2}\delta(x-a)\psi = k^2\psi$

$\psi'\Big|_{a-\epsilon}^{a+\epsilon} + \frac{2m\alpha}{\hbar^2}\psi(a) = 0 \Rightarrow$

$-kB - kA\cosh(ka) + \frac{2m\alpha}{\hbar^2}B, \quad A\sinh(ka) = B$

$\left(-k - k\coth(ka) + \frac{2m\alpha}{\hbar^2}\right)B = 0$

$ka(1+\coth(ka)) = 2\frac{\alpha m a}{\hbar^2}$

$\geq 1$ for $ka \geq 0$; 1-to-1 $\Rightarrow \begin{cases} 1 \text{ state for } \frac{\alpha m a}{\hbar^2} \geq \frac{1}{2} \\ 0 \text{ states otherwise} \end{cases}$

force is attractive. (to $\delta$ wall) (repulsive to $x<0$ wall)

since it allows a bound state which would otherwise be impossible

since it prevents bound state for small $\alpha$

QM1-4 $s = \frac{1}{2} \quad \left(S^2 = \frac{3}{4}\hbar^2\right), \quad \vec{\sigma}\cdot\hat{n} = \begin{pmatrix} n_3 & n_1 - n_2 i \\ n_1 + n_2 i & -n_3 \end{pmatrix}$

$\langle \vec{\sigma}\cdot\hat{n}\rangle = (1\ 0)\begin{pmatrix} n_3 & n_1 - in_2 \\ n_1 + in_2 & -n_3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = (1\ 0)\begin{pmatrix} n_3 \\ n_1 + in_2 \end{pmatrix} = n_3 = \hat{n}\cdot\hat{z}$

to obtain $|+\hat{n}\rangle$ rotate $|+\hat{z}\rangle$ by $\frac{\pi}{4}$ about $\hat{x}$ (say).

$|+\hat{n}\rangle = e^{-iS_x/\hbar \cdot \frac{\pi}{4}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = e^{-i\pi/8\,\sigma_x}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \left(\cos\left(\frac{\pi}{8}\right) - i\sigma_x \sin\left(\frac{\pi}{8}\right)\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} \cos\frac{\pi}{8} & -i\sin\pi/8 \\ -i\sin\pi/8 & -\cos\pi/8 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\pi/8 \\ -i\sin\pi/8 \end{pmatrix}$

$|\langle +\hat{n}|+\hat{z}\rangle|^2 = \left|(\cos\pi/8 \quad i\sin\pi/8)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right|^2 = |\cos\pi/8|^2 = \cos^2\pi/8$

$S_z = +\frac{\hbar}{2}$: $\cos^2\pi/8$. $\quad S_z = -\frac{\hbar}{2}$: $\sin^2\left(\frac{\pi}{8}\right)$

2010 QM1-5 vibrational levels $\sim \hbar\omega = \hbar\sqrt{\frac{2k}{m}} \sim \hbar\sqrt{\frac{k}{m}}$

rotational levels $\sim \frac{L^2}{2mr^2} \sim \frac{\hbar^2}{mr_0^2}$

so $\frac{\hbar^2}{mr_0^2} << \hbar\sqrt{\frac{k}{m}}$

Rotational hamiltonian $H_1 = \frac{l(l+1)\hbar^2}{2mr_0^2}$

$\omega = \frac{2\hbar^2}{2mr_0^2\hbar} = \frac{\hbar}{mr_0^2}$ for $l=1 \to l=0$

SM1 $Q = \sum_{N=0}^{\infty} z^N \sum_{\{s\}} e^{-\beta\epsilon_s} = \sum_{n_s=0 \text{ or } 1} z^{\sum_s n_s} e^{-\beta\sum_s n_s\epsilon_s} = \sum_{n_s=0,1} \prod_s (ze^{-\beta\epsilon_s})^{n_s}$

$= \prod_s (1+ze^{-\beta\epsilon_s})$

$PV = kT\log Q \Rightarrow \Phi = -kT\sum_s \log(1+ze^{-\beta\epsilon_s})$ , $\beta = \frac{1}{kT}$

SM2 $\epsilon_l = \frac{l(l+1)\hbar^2}{2I}$

$Q_1 = \sum_{l=0}^{\infty}(2l+1)e^{-\beta\epsilon_l} = \sum_{l=0}^{\infty}(2l+1)e^{-\frac{l(l+1)\hbar^2}{2IkT}}$

$U = \frac{-\partial Q_1/\partial\beta}{Q_1} = \frac{\sum_{l=0}^{\infty}(2l+1)\frac{l(l+1)\hbar^2}{2I}e^{-\frac{l(l+1)\hbar^2}{2IkT}}}{\sum_{l=0}^{\infty}(2l+1)e^{-l(l+1)\hbar^2/2IkT}}$

$T\to\infty \Rightarrow$ approx with smallest nonzero $l$ term

$U \to \frac{3\cdot 2\hbar^2/2I\, e^{-2\hbar^2/2IkT}}{1+3e^{-2\hbar^2/2IkT}} = \frac{\hbar^2}{I}\cdot\frac{3}{3+e^{\hbar^2/IkT}} \longrightarrow \frac{3\hbar^2}{I}e^{-\hbar^2/IkT}$

with nuclear spin, triplet $\Rightarrow$ antisymmetric $l$ state $\Rightarrow$ $l$ odd

singlet $\Rightarrow$ sym $l$ state $\Rightarrow$ $l$ even

then $Q_1 = 3\sum_{l\text{ odd}}(2l+1)e^{-\frac{l(l+1)\hbar^2}{2IkT}} + \sum_{l\text{ even}}(2l+1)e^{-\frac{l(l+1)\hbar^2}{2IkT}}$

$$dH = TdS + VdP \qquad dG = -SdT + VdP$$
$$dU = TdS - PdV \qquad dA = -SdT - PdV$$

SM3 $$\left.\frac{\partial U}{\partial V}\right|_T = \left.\frac{\partial U}{\partial V}\right|_S + \left.\frac{\partial U}{\partial S}\right|_V \left.\frac{\partial S}{\partial V}\right|_T = -P + T\left.\frac{\partial S}{\partial V}\right|_T$$

but $$\left.\frac{\partial S}{\partial V}\right|_T = -\left.\frac{\partial}{\partial V}\right|_T \left.\frac{\partial}{\partial T}\right|_V A = -\left.\frac{\partial}{\partial T}\right|_V \left.\frac{\partial}{\partial V}\right|_T A = \left.\frac{\partial P}{\partial T}\right|_V$$

So $$\left.\frac{\partial U}{\partial V}\right|_T = -P + T\left.\frac{\partial P}{\partial T}\right|_V = \frac{-RT}{V-B} + \frac{a}{V^2} + T\frac{R}{V-B} = \frac{a}{V^2}$$

SM4 $$\langle (E-\langle E\rangle)^2\rangle = \langle E^2 - 2E\langle E\rangle + \langle E\rangle^2\rangle = \langle E^2\rangle - \langle E\rangle^2$$

$$U = \frac{\sum_r E_r e^{-\beta E_r}}{\sum_r e^{-\beta E_r}}, \quad \frac{\partial U}{\partial \beta} = \frac{-\sum_r E_r^2 e^{-\beta E_r}}{\sum_r e^{-\beta E_r}} + \frac{\left(\sum_r E_r e^{-\beta E_r}\right)^2}{\left(\sum_r e^{-\beta E_r}\right)^2} = -\langle E^2\rangle + \langle E\rangle^2$$

$$= \frac{\partial T}{\partial \beta}\frac{\partial U}{\partial T} = -kT^2\frac{\partial U}{\partial T} = -kT^2 C_V$$

$$\Rightarrow \langle (E-\langle E\rangle)^2\rangle = kT^2 C_V$$

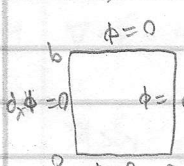
$$\frac{\langle (E-\langle E\rangle)^2\rangle}{\langle E^2\rangle} = \frac{kT^2 C_V}{U^2 + kT^2 C_V} \overset{N\gg 1}{\simeq} \frac{kT^2 C_V}{U^2}, \quad T\to\infty \Rightarrow U \sim C_V T$$

$$\overset{T\to\infty}{\simeq} \frac{k}{C_V}$$

SM5 $$F\frac{\hbar}{mc} = \frac{\hbar a}{2\pi ck}\, 2\pi k \Rightarrow F = ma$$

$$E = Mc^2 = \frac{4\pi r^2 c^3}{G_N \hbar}\frac{1}{2}\frac{\hbar a}{2\pi c} \Rightarrow \frac{G_N M}{r^2} = a = \frac{F}{m} \Rightarrow F = \frac{G_N M m}{r^2}$$

EM1-1




$$\nabla^2\phi = 0 = \partial_x^2\phi + \partial_y^2\phi, \quad \phi = X(x)Y(y)$$
$$= X''Y + XY'' \Rightarrow \underbrace{\frac{X''}{X}}_{k^2} + \underbrace{\frac{Y''}{Y}}_{-k^2} = 0$$

$$Y \propto \sin\left(\frac{n\pi y}{b}\right), \quad k = \frac{n\pi}{b} \Rightarrow X \propto \cosh(kx) = \cosh\left(\frac{n\pi x}{b}\right) \quad \text{since } X'(0) = 0$$

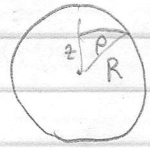
$$\phi(x,y) = \sum_n c_n \cosh\left(\frac{n\pi x}{b}\right)\sin\left(\frac{n\pi y}{b}\right), \quad \phi(a,y) = \sum_n c_n\cosh\left(\frac{n\pi a}{b}\right)\sin\left(\frac{n\pi y}{b}\right) = \phi_0$$

$$\Rightarrow c_n\cosh\left(\frac{n\pi a}{b}\right)\frac{b}{2} = \phi_0\int_0^b \sin\left(\frac{n\pi y}{b}\right)dy = -\phi_0\frac{b}{n\pi}\cos\left(\frac{n\pi y}{b}\right)\Big|_0^b = \phi_0\frac{b}{n\pi}\begin{cases}2, & n \text{ odd}\\ 0, & n \text{ even}\end{cases}$$

$$\phi(x,y) = \frac{2}{\pi}\phi_0\sum_{n=0}^{\infty}\frac{2}{2n+1}\frac{\cosh\left((2n+1)\pi\frac{x}{b}\right)}{\cosh\left((2n+1)\pi\frac{a}{b}\right)}\sin\left((2n+1)\pi\frac{y}{b}\right)$$

only odd terms due to symmetry about $y = \frac{b}{2}$.

2010 EM1-2




$$\rho^2 + z^2 = R^2, \quad v = \rho\omega, \quad K = \sigma v$$

$$\Rightarrow \vec{K} = \sigma\omega\sqrt{R^2 - z^2}\,\hat{\varphi}, \quad z = R\cos\theta$$

$$\vec{K} = \sigma\omega R\sin\theta\,\hat{\varphi}$$

$$\hat{r}\times(\vec{B}_{out} - \vec{B}_{in}) = \sigma\omega R\sin\theta\,\hat{\varphi} \Rightarrow \vec{B}_{out} - \vec{B}_{in} = -\sigma\omega R\sin\theta\,\hat{\theta}$$

$$\vec{B} = -\nabla\phi_m = -\frac{\partial\phi_m}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial\phi_m}{\partial\theta}\hat{\theta} - \frac{1}{r\sin\theta}\frac{\partial\phi_m}{\partial\varphi}\hat{\varphi}$$

$$\Rightarrow -\sigma\omega R\sin\theta = -\frac{1}{R}\left(\left.\frac{\partial\phi_m}{\partial\theta}\right|_{out} - \left.\frac{\partial\phi_m}{\partial\theta}\right|_{in}\right)$$

$$\phi_m = \begin{cases} \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta), & r < R \\ \sum_{l=0}^{\infty} B_l r^{-l-1} P_l(\cos\theta), & r > R \end{cases}$$

$$\therefore \frac{\partial\phi_m}{\partial\theta} \propto \sin\theta \Rightarrow \text{only } l = 1$$

$$-\sigma\omega R\sin\theta = -\frac{1}{R}\left(-B_1 R^{-2}\sin\theta + A_1 R\sin\theta\right)$$

$$A_1 R - B_1 R^{-2} = \sigma\omega R^2$$

but also $\frac{\partial\Phi_m}{\partial r}$ cont. $\Rightarrow A_1\cos\theta = -2B_1 R^{-3}\cos\theta$

$$\Rightarrow B_1 = -\frac{1}{2}R^3 A_1$$

$$A_1\left(1+\frac{1}{2}\right)R = \sigma\omega R^2 \Rightarrow A_1 = \frac{2}{3}\sigma\omega R, \quad B_1 = -\frac{1}{3}\sigma\omega R^4$$

$$\phi_m = \begin{cases} \frac{2}{3}\sigma\omega R r\cos\theta, & r < R \\ -\frac{1}{3}\sigma\omega\frac{R^4}{r^2}\cos\theta, & r > R \end{cases}$$

$$\vec{B} = -\nabla\phi_m = \begin{cases} -\frac{2}{3}\sigma\omega R\cos\theta\,\hat{r} + \frac{2}{3}\sigma\omega R\sin\theta\,\hat{\theta}, & r < R \\ -\frac{2}{3}\sigma\omega\frac{R^4}{r^3}\cos\theta\,\hat{r} - \frac{1}{3}\sigma\omega\frac{R^4}{r^3}\sin\theta\,\hat{\theta}, & r > R \end{cases}$$

EM1-3

$a \cdot q$

$\cdot -2q$

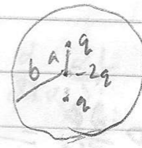
$-a \cdot q$

$$\phi = \frac{-2q}{r} + \frac{q}{|\vec{r} - a\hat{z}|} + \frac{q}{|\vec{r} + a\hat{z}|} \qquad (1+x)^{-1/2} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 \ldots$$

$$\frac{1}{|\vec{r} \pm a\hat{z}|} = \left((\vec{r} \pm a\hat{z})^2\right)^{-1/2} = \left(r^2 \pm 2ar\cos\theta + a^2\right)^{-1/2} = \frac{1}{r}\left(1 \pm \frac{2a}{r}\cos\theta + \frac{a^2}{r^2}\right)^{-1/2}$$

$$\simeq \frac{1}{r}\left(1 \mp \frac{a}{r}\cos\theta - \frac{1}{2}\frac{a^2}{r^2} + \frac{3}{2}\frac{a^2}{r^2}\cos^2\theta\right)$$

$$\phi = \frac{q}{r}\left(-\frac{a^2}{r^2} + 3\frac{a^2}{r^2}\cos^2\theta\right) = \frac{qa^2}{r^3}\left(3\cos^2\theta - 1\right)$$



images: $Q = \frac{-r}{b} q$ at $z = \pm \frac{b^2}{a}$ and $Q = \frac{c}{b} 2q$ at (say) $z = c$ as $c \to \infty$

$\Rightarrow Q = -\frac{b}{a} q$ at $z = \pm \frac{b^2}{a}$

$$\phi_{image} = -\frac{b}{a} q \left( \frac{1}{|\vec{r} - \frac{b^2}{a}\hat{z}|} + \frac{1}{|\vec{r} + \frac{b^2}{a}\hat{z}|} \right) + \underbrace{\lim_{c \to \infty} 2q \frac{c}{b} \frac{1}{|\vec{r} - c\hat{z}|}}_{\frac{2q}{b}}$$

$$= \frac{2q}{b} - q \left( \frac{1}{|b\hat{z} - \frac{a}{b}\vec{r}|} + \frac{1}{|b\hat{z} + \frac{a}{b}\vec{r}|} \right)$$

$$= \frac{q}{b} \left[ 2 - \left( (\hat{z} - \tfrac{a}{b^2}\vec{r})^2 \right)^{-1/2} + \left( (\hat{z} + \tfrac{a}{b^2}\vec{r})^2 \right)^{-1/2} \right]$$

$$= \frac{q}{b} \left[ 2 - \underbrace{\left( 1 - \frac{2a}{b^2} r\cos\theta + \frac{a^2 r^2}{b^4} \right)^{-1/2}}_{1 + \frac{a}{b^2} r\cos\theta - \frac{1}{2}\frac{a^2 r^2}{b^4} + \frac{3}{2}\frac{a^2}{b^4} r^2 \cos^2\theta} + \left( 1 + \frac{2a}{b^2} r\cos\theta + \frac{a^2 r^2}{b^4} \right)^{-1/2} \right]$$

$$= \frac{q}{b} \left[ -\frac{a^2 r^2}{b^4} + 3\frac{a^2}{b^4} r^2 \cos^2\theta \right] = \frac{q a^2 r^2}{b^5} (3\cos^2\theta - 1)$$

$$\phi = \frac{q a^2}{r^3} (3\cos^2\theta - 1) \left( 1 + \left( \frac{r}{b} \right)^5 \right)$$

EM1-4 $\nabla \cdot \vec{E} = 4\pi\rho \qquad \nabla \cdot \vec{B} = 0 \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla \times \vec{B} = \frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$

$\nabla \cdot \vec{B} = \partial^i B_i = \underbrace{\epsilon_{ijk}}_{\text{antysymmetric}} \overbrace{\partial^i \partial^j}^{\text{symmetric}} A^k = 0$

$(\nabla \times \vec{E})_i = \epsilon_{ijk} \partial^j E^k = \epsilon_{ijk} \partial^j (-\partial^k \phi + \partial_0 A^k) = -\underbrace{\epsilon_{ijk} \partial^j \partial^k \phi}_{0} - \partial_0 \underbrace{\epsilon_{ijk} \partial^j A^k}_{B_i} = \left( -\frac{\partial \vec{B}}{\partial t} \right)_i$

$(\nabla \times (\nabla \times \vec{A}))_i = \epsilon_{ijk} \partial^j \epsilon_{klm} \partial^l A^m = \epsilon_{ijk} \epsilon_{lmk} \partial^j \partial^l A^m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \partial^j \partial^l A^m$

$= \partial_j \partial_i A^j - \partial_j \partial^j A_i = (\nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A})_i$

$\vec{A} \to \vec{A} + \nabla \Lambda$ leaves $\vec{B}$ invariant since $\nabla \times (\nabla \Lambda) = 0$

$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t} \to \vec{E} + \nabla \frac{\partial \Lambda}{\partial t} - \frac{\partial}{\partial t} \nabla \Lambda = \vec{E}$

$$\nabla\cdot\vec{E} = \nabla\cdot\left(-\nabla\phi - \frac{\partial\vec{A}}{\partial t}\right) = -\nabla^2\phi - \frac{\partial}{\partial t}\nabla\cdot\vec{A} = -\nabla^2\phi = 4\pi\rho$$

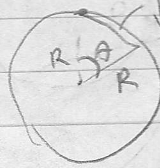
$$\nabla\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \nabla\times(\nabla\times\vec{A}) + \frac{1}{c}\frac{\partial}{\partial t}\nabla\phi = \nabla(\nabla\cdot\vec{A})^{\to 0} - \nabla^2\vec{A} + \frac{1}{c}\nabla\frac{\partial\phi}{\partial t}^{\to 0} = -\nabla^2\vec{A} = \frac{4\pi}{c}$$

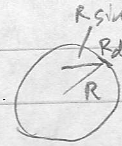
$$\nabla^2\phi = -4\pi\rho, \quad \nabla^2\vec{A} = -\frac{4\pi}{c}\vec{J}$$

EM1-5 charge density $\rho_c = \frac{Q}{4\pi R^2}$

force density $\vec{f} = \rho_c \int \frac{dQ}{r^2}\hat{r}$, $\quad dQ = \rho_c\, dA = \rho_c \cdot 2\pi R\sin\theta \cdot R\,d\theta$

$\sqrt{R^2+R^2-2RR\cos\theta} = \sqrt{2}R\sqrt{1-\cos\theta} = r$




$$\vec{f} = \rho_c^2\, 2\pi R^2 \int_0^\pi \sin\theta\, d\theta\, \frac{R\hat{z} - R\hat{z}\cos\theta + (x, y \text{ which cancel})}{2^{3/2}R^3(1-\cos\theta)^{3/2}}$$

$$= \rho_c^2 \frac{\pi}{\sqrt{2}}\hat{z}\int_0^\pi \frac{\sin\theta\, d\theta}{(1-\cos\theta)^{3/2}} = \rho_c^2\frac{\pi}{\sqrt{2}}\hat{z}\int_{-1}^{1}(1-\mu)^{-3/2}d\mu$$

$$= \rho_c^2\frac{\pi}{\sqrt{2}}\hat{z}\, 2(1-\mu)^{-1/2}\Big|_{-1}^{1} =$$

CM1 $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$ , $t \to \beta t$, $q_i \to \alpha q_i$

$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} + \frac{\partial V}{\partial q_i} = 0$ , $V \propto q_i^n \Rightarrow \frac{\partial V}{\partial q_i} \propto n q_i^{n-1}$

$T \propto \dot{q}_i^2 \Rightarrow \frac{\partial T}{\partial \dot{q}_i} \propto 2\dot{q}_i$

$q_i \to \alpha q_i \Rightarrow \frac{\partial V}{\partial q_i} \to \alpha^{n-1}\frac{\partial V}{\partial q_i}$

$\dot{q}_i \to \frac{\alpha}{\beta}\dot{q}_i \Rightarrow \frac{\partial T}{\partial \dot{q}_i} \to \frac{\alpha}{\beta}\frac{\partial T}{\partial \dot{q}_i} \Rightarrow \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} \to \frac{\alpha}{\beta^2}\frac{\partial T}{\partial \dot{q}_i}$

so EOM unchanged if $\frac{\alpha}{\beta^2} = \alpha^{n-1} \Rightarrow \beta^2 = \alpha^{2-n} \Rightarrow \beta = \alpha^{1-\frac{n}{2}}$

so consider path $q_i(t)$, then $\alpha q_i(\beta t)$ also satisfies EOM.

i.e. $\frac{l'}{l} = \alpha$, $\frac{t'}{t} = \beta \Rightarrow \frac{t'}{t} = \left(\frac{l'}{l}\right)^{1-\frac{n}{2}}$

HO: $n=2 \Rightarrow \frac{t'}{t} = \left(\frac{l'}{l}\right)^{1-\frac{2}{2}} = 1 \Rightarrow$ period indep. of length scale

Free fall: $n=1 \Rightarrow \frac{t'}{t} = \left(\frac{l'}{l}\right)^{1-\frac{1}{2}} = \left(\frac{l'}{l}\right)^{1/2}$

CM2 $f(t)\left[\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{1}{2}m\omega^2 q^2\right] + \frac{\partial S}{\partial t} = 0$

Separate $S = S_q(q) + S_t(t)$, $\underbrace{\frac{1}{2m}\left(\frac{\partial S_q}{\partial q}\right)^2 + \frac{1}{2}m\omega^2 q^2}_{k} + \underbrace{\frac{1}{f(t)}\frac{\partial S_t}{\partial t}}_{-k} = 0$

$\frac{\partial S_t}{\partial t} = -kf(t) = -k\frac{dg}{dt} \Rightarrow S_t(t) = -k g(t) + \alpha_0$

$\left(\frac{\partial S_q}{\partial q}\right)^2 = 2mk - m^2\omega^2 q^2 \Rightarrow \frac{\partial S_q}{\partial q} = \sqrt{2mk - m^2\omega^2 q^2} = \sqrt{2mk}\sqrt{1 - \frac{m\omega^2}{2k}q^2}$

$S_q = \sqrt{2mk}\int dq\sqrt{1 - \frac{m\omega^2}{2k}q^2}$ , $\sqrt{\frac{m\omega^2}{2k}}q = x$, $x = \sin\theta$, $dx = \cos\theta\, d\theta$

$= \sqrt{2mk}\sqrt{\frac{2k}{m\omega^2}}\int dx\sqrt{1-x^2} = \frac{2k}{\omega}\int \cos^2\theta\, d\theta = \frac{2k}{\omega}\int \frac{1}{2}(1+\cos 2\theta)\, d\theta$

$= \frac{k}{\omega}\left(\theta + \frac{1}{2}\sin 2\theta\right) = \frac{k}{\omega}\left[\sin^{-1}\left(\sqrt{\frac{m\omega^2}{2k}}q\right) + \sqrt{\frac{m\omega^2}{2k}}q\sqrt{1 - \frac{m\omega^2}{2k}q^2}\right]$

$$y = \sin^{-1} x \quad x = \sin y\,,\ 1 = \cos y \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$S = \alpha_0 - \alpha g(t) + \frac{\alpha}{\omega}\left[\sin^{-1}\left(\sqrt{\frac{m\omega^2}{2\alpha}}q\right) + \sqrt{\frac{m\omega^2}{2\alpha}}q\sqrt{1-\frac{m\omega^2}{2\alpha}q^2}\right] \qquad \alpha \equiv k$$

$$\beta = \frac{\partial S}{\partial \alpha} = -g(t) + \frac{1}{\omega}\left[\sin^{-1}\left(\sqrt{\frac{m\omega^2}{2\alpha}}q\right) + \sqrt{\frac{m\omega^2}{2\alpha}}q\sqrt{1-\frac{m\omega^2}{2\alpha}q^2}\right]$$

$$+\frac{\alpha}{\omega}\left[\frac{1}{\sqrt{1-\frac{m\omega^2}{2\alpha}q^2}}\sqrt{\frac{m\omega^2}{2}}\,q\frac{-1}{2\alpha^{3/2}} + \sqrt{\frac{m\omega^2}{2}}q\frac{-1}{2\alpha^{3/2}}\sqrt{1-\frac{m\omega^2}{2\alpha}q^2}\right.$$

$$\left. + \sqrt{\frac{m\omega^2}{2\alpha}}q\frac{1/2}{\sqrt{1-\frac{m\omega^2}{2\alpha}q^2}}\frac{m\omega^2}{2\alpha^2}q^2\right]$$

$$\beta = -g(t) + \frac{1}{\omega}\left[\sin^{-1}\left(\sqrt{\frac{m\omega^2}{2\alpha}}q\right) + \frac{1}{2}\sqrt{\frac{m\omega^2}{2\alpha}}q\sqrt{1-\frac{m\omega^2}{2\alpha}q^2}\right.$$

$$\left. -\frac{1}{2}\sqrt{\frac{m\omega^2}{2\alpha}}q\sqrt{1-\frac{m\omega^2}{2\alpha}q^2}\right]$$

$$\beta = -g(t) + \frac{1}{\omega}\sin^{-1}\left(\sqrt{\frac{m\omega^2}{2\alpha}}q\right)$$

$$q = \sqrt{\frac{2\alpha}{m\omega^2}}\sin\left[\omega(g(t)+\beta)\right]$$ , $\alpha$ is the energy

CM3 $V = a^2x^4 - 2b^2x^2$, $\frac{dV}{dx} = 4a^2x^3 - 4b^2x = 4a^2\left(x^2 - \left(\frac{b}{a}\right)^2\right)x = 0$

$$\Rightarrow x = 0,\ x = \pm\frac{b}{a}$$

$\frac{d^2V}{dx^2} = 12a^2x^2 - 4b^2$ $< 0$ for $x=0$ → unstable

$> 0$ for $x = \pm\frac{b}{a}$ → stable

consider $x = \pm\frac{b}{a} + q$, $q$ small

$$\ddot{q} = \frac{F}{m} = -\frac{1}{m}\frac{\partial V}{\partial q} = -\frac{4a^2}{m}\left(\pm 2\frac{b}{a}q\right)\left(\pm\frac{b}{a} + q\right) = -\frac{4a^2}{m}\cdot 2\frac{b^2}{a^2}q = -\frac{8b^2}{m}q$$

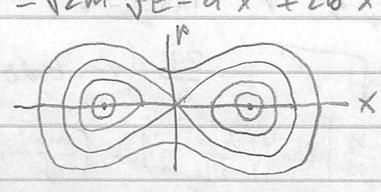
$$= -\omega^2 q \Rightarrow \omega = \sqrt{\frac{8}{m}}\,b$$

consider $x$ small. $\ddot{x} = -\frac{1}{m}4a^2(-1)\frac{b^2}{a^2}x = \frac{4b^2}{m}x$

$$\Rightarrow x \propto e^{\frac{2b}{\sqrt{m}}t}$$

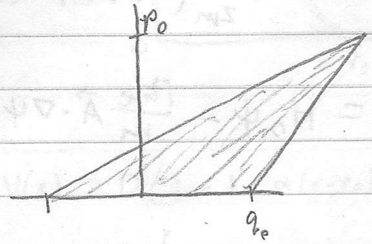
$L = \frac{1}{2}m\dot{x}^2 - a^2x^4 + 2b^2x^2$, $p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$, $H = \frac{1}{2}m\dot{x}^2 + a^2x^4 - 2b^2x^2 = \frac{p^2}{2m} + a^2x^4 - 2b^2x^2$

const. energy: $p = \sqrt{2m}\sqrt{E - a^2x^4 + 2b^2x^2}$



CM 4 free particles: p const, $\Delta q$ same for particles at given p, and $\Delta q \propto p$.

e.g.




So dist. remains a triangle with same base and height $\Rightarrow$ same area.

$p(t) = p_0, \quad q(t) = q_0 + \frac{p_0}{m} t$

write $M = \begin{pmatrix} \frac{\partial q}{\partial q_0} & \frac{\partial q}{\partial p_0} \\ \frac{\partial p}{\partial q_0} & \frac{\partial p}{\partial p_0} \end{pmatrix} = \begin{pmatrix} 1 & t/m \\ 0 & 1 \end{pmatrix}$

$MJ\tilde{M} = \begin{pmatrix} 1 & t/m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t/m & 1 \end{pmatrix} = \begin{pmatrix} 1 & t/m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t/m & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = J$

$\Rightarrow M$ is canonical

CM 5 $T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{m}{2} \Big( (-(a + b\sin\theta)\sin\varphi\,\dot{\varphi} + b\cos\theta\,\dot{\theta}\cos\varphi)^2 + ((a+b\sin\theta)\cos\varphi\,\dot{\varphi} + b\cos\theta\,\dot{\theta}\sin\varphi)^2 + b^2\sin^2\theta\,\dot{\theta}^2 \Big)$

$= \frac{m}{2} \left[ (a + b\sin\theta)^2 \dot{\varphi}^2 + b^2 \dot{\theta}^2 \right]$

$V = mgz = mgb\cos\theta$

$L = \frac{m}{2} \left[ (a+b\sin\theta)^2 \dot{\varphi}^2 + b^2 \dot{\theta}^2 \right] - mgb\cos\theta$

$\frac{\partial L}{\partial \varphi} = 0 \Rightarrow p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = m(a+b\sin\theta)^2 \dot{\varphi}$ conserved

$\frac{\partial L}{\partial t} = 0 \Rightarrow E = \frac{m}{2} \left[ (a+b\sin\theta)^2 \dot{\varphi}^2 + b^2 \dot{\theta}^2 \right] + mgb\cos\theta$ conserved

$E = \underbrace{\frac{p_\varphi^2}{2m(a+b\sin\theta)^2} + mgb\cos\theta}_{V_{eff}} + \frac{1}{2} m b^2 \dot{\theta}^2$

$\dot{\varphi} = \Omega \Rightarrow p_\varphi = m(a + b\sin\theta_c)\Omega$

eq. $\Rightarrow \dot{\theta} = 0 \Rightarrow E = \frac{m}{2} (a + b\sin\theta_c)^2 \Omega^2 + mgb\cos\theta_c$

2010 QMII-1 $H = \frac{1}{2m}(\vec{p} + e\vec{A})^2 + V(r)$

$$= \frac{1}{2m}(-i\hbar\nabla + e\vec{A})^2 + V(r) = \underbrace{\frac{-\hbar^2}{2m}\nabla^2 + V(r)}_{H_0} + \frac{1}{2m}(-i\hbar\nabla\cdot e\vec{A} - i\hbar e A\cdot\nabla) + \mathcal{O}(\vec{A}^2)$$

$$H\psi = H_0\psi - \frac{i\hbar e}{2m}\left(\underbrace{\nabla\cdot(\vec{A}\psi)}_{(\nabla\cdot\vec{A})\psi + \vec{A}\cdot\nabla\psi,\ \nabla\cdot\vec{A}=0} + \vec{A}\cdot\nabla\psi\right) = H_0\psi - \frac{i\hbar e}{m}\vec{A}\cdot\nabla\psi$$

$$V = -\frac{i\hbar e}{m}\cos\omega t\ \vec{A}_0\cdot\nabla$$

$$V_{nm} = \int d^3r\ \psi_n^* V \psi_m \quad \text{where } \psi_n, \psi_m \text{ indep. of time}$$

$$(\psi_n(t) = \psi_n e^{-iE_n/\hbar\, t}, \text{ etc.})$$

$$= \text{const} \cdot \cos\omega t$$

$$c_{nm} = -\frac{i}{\hbar}\int_{t_0}^{t} e^{i\omega_{nm}t'} V_{nm}(t')\,dt' = \text{const}\cdot\int_{t_0}^{t} e^{i\omega_{nm}t'}\cos\omega t'\,dt'$$

$$\left(\omega_{nm} = \frac{E_n - E_m}{\hbar}\right)$$

$$c_{nm} = \text{const}\cdot\int_0^t \left[e^{i(\omega_{nm}+\omega)t'} + e^{i(\omega_{nm}-\omega)t'}\right]dt$$

$$= \text{const}\cdot\left[\frac{e^{i(\omega_{nm}+\omega)t} - 1}{\omega_{nm}+\omega} + \frac{e^{i(\omega_{nm}-\omega)t} - 1}{\omega_{nm}-\omega}\right]$$

$$|c_{nm}|^2 = \text{const}\cdot\left[\frac{2 - 2\cos(\omega_{nm}+\omega)t}{(\omega_{nm}+\omega)^2} + \frac{2 - 2\cos(\omega_{nm}-\omega)t}{(\omega_{nm}-\omega)^2} + \frac{(e^{i(\omega_{nm}+\omega)t} - 1)(e^{i(\omega_{nm}+\omega)t} - 1) + (e^{i(-\omega_{nm}-\omega)t} - 1)(e^{i(\omega_{nm}-\omega)t} - 1)}{(\omega_{nm}+\omega)(\omega_{nm}-\omega)}\right]$$

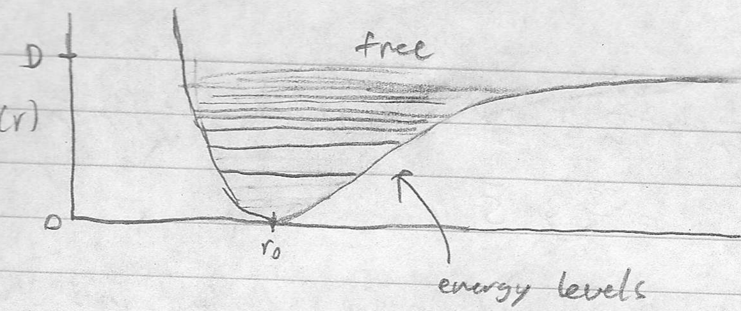
$$= 2\cos 2\omega t + 2 - 2\cos(\omega_{nm}-\omega)t - 2\cos(\omega_{nm}+\omega)t$$

$$= \text{const}\cdot\left[\frac{1 - \cos(\omega_{nm}+\omega)t}{(\omega_{nm}+\omega)^2} + \frac{1 - \cos(\omega_{nm}-\omega)t}{(\omega_{nm}-\omega)^2} + \frac{1 + \cos 2\omega t - \cos(\omega_{nm}-\omega)t - \cos(\omega_{nm}+\omega)t}{(\omega_{nm}+\omega)(\omega_{nm}-\omega)}\right]$$

$$\omega_{mn} = -\omega_{nm} \simeq \omega \Rightarrow |c_{nm}|^2 \simeq \text{const}\cdot\left[\frac{1 - \cos\Delta\omega t}{\Delta\omega^2} + \underbrace{\frac{1 - \cos 2\omega t}{(2\omega)^2}}_{\text{small}} + \underbrace{\frac{1 - \cos\Delta\omega t}{2\omega\Delta\omega}}_{\text{small}}\right]$$

$$|c_{nm}|^2 \simeq \text{const}\cdot\left[\frac{\frac{1}{2}(\Delta\omega t)^2}{\Delta\omega^2}\right] = \text{const}\cdot t^2 \qquad (\Delta\omega \equiv \omega_{mn} - \omega)$$

QMII-2




D is the binding energy.

$r_0$ is the most likely / most stable distance between nuclei.

$$V(r) = D\left(1 - 1 + \alpha(r-r_0) - \frac{1}{2}\alpha^2(r-r_0)^2 + \dots\right)^2$$

$$\simeq D\left(\alpha(r-r_0) - \frac{1}{2}\alpha^2(r-r_0)^2 + \frac{1}{6}\alpha^3(r-r_0)^3\right)^2 = D\alpha^2x^2\left(1-\frac{1}{2}\alpha x + \frac{1}{6}\alpha^2x^2\right)^2, \quad x = r-r_0$$

$$\simeq D\alpha^2x^2\left(1 - \alpha x + \frac{1}{3}\alpha^2x^2 + \frac{1}{4}\alpha^2x^2\right) = D\alpha^2x^2\left(1-\alpha x + \frac{7}{12}\alpha^2x^2\right)$$

$$\simeq \underbrace{D\alpha^2x^2} - D\alpha^3x^3 + \frac{7}{12}D\alpha^4x^4$$

harmonic oscillator with $m\omega^2 = D\alpha^2$

$$\Rightarrow \quad E_0 = \frac{1}{2}\hbar\omega = \frac{1}{2}\hbar\sqrt{\frac{D}{m}}\,\alpha \qquad (m = \text{reduced mass})$$

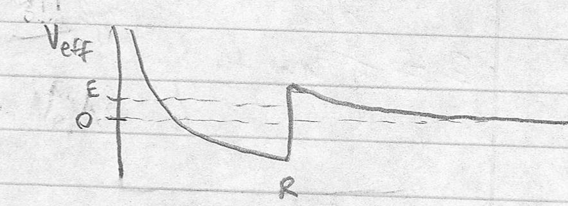
$$\psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}, \quad x = r-r_0, \quad \omega = \sqrt{\frac{D}{m}}\,\alpha$$

$$E_0^{(1)} = \int \psi_0^* \left(\cancelto{0}{-D\alpha^3x^3} + \frac{7}{12}D\alpha^4x^4\right)\psi_0 \, dx$$

$$= \sqrt{\frac{m\omega}{\pi\hbar}}\,\frac{7}{12}D\alpha^4 \underbrace{\int x^4 e^{-\frac{m\omega}{\hbar}x^2}dx}_{2\frac{3!!}{2\left(2\frac{m\omega}{\hbar}\right)^2}} = \qquad , \quad 3!! = 3$$

QMII-3

$$E = \underbrace{\underbrace{\frac{l(l+1)\hbar^2}{2mr^2}} + \frac{p_r^2}{2m} \underbrace{- V\theta(R-r)}}_{V_{eff}}$$




$$\tau \sim \frac{2\pi\hbar}{\Delta E} \quad \text{where} \quad \Delta E = V_{eff}(R+\epsilon) - E = \frac{l(l+1)\hbar^2}{2mR^2} - E$$

2010 QMH-4 $H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{p_3^2}{2m} + \frac{1}{2} m\omega^2 (r_1^2 + r_2^2 + r_3^2) - \beta \vec{\mu}_1 \cdot \vec{\mu}_2 - \beta \vec{\mu}_1 \cdot \vec{\mu}_3 - \beta \vec{\mu}_2 \cdot \vec{\mu}_3$

where $\vec{\mu}_i = \frac{2}{\hbar} \mu \vec{S}_i$

spatial part is separable, so states look like sums of

$\phi_A(\vec{x}_1)\phi_B(\vec{x}_2)\phi_C(\vec{x}_3)\chi(s_1, s_2, s_3)$

for lowest energy spatial state, two of the $\phi$ are the ground state $\phi_0$ while the third is one of the 3 1st excited states $\phi_1^{(i)}$. ← in x, y, or z

So spatial $\phi$ must be symmetric $\phi_0\phi_0\phi_1 + \phi_0\phi_1\phi_0 + \phi_1\phi_0\phi_0$.

$\Rightarrow \chi$ antisymmetric: impossible.

Thus take spatial state $\phi_0 \phi_1^{(i)} \phi_1^{(j)}$ antisymmetrized.

3 possible $i, j$ different $\quad E_\phi = \frac{3}{2}\hbar\omega + 2\hbar\omega + 2\hbar\omega = \frac{11}{2}\hbar\omega$

$\Rightarrow \chi$ symmetric, 4 states: $\downarrow\downarrow\downarrow$ : $-3\beta\mu^2$

$\uparrow\uparrow\uparrow$ : $-3\beta\mu^2$

$\uparrow\downarrow\downarrow + \downarrow\uparrow\downarrow + \downarrow\downarrow\uparrow$ : $+\beta\mu^2$

$+\beta\mu^2$

$$E = \frac{11}{2}\hbar\omega + \begin{cases} -3\beta\mu^2 & \text{deg. } 6 \\ \beta\mu^2 & \text{deg. } 6 \end{cases}$$