UNIVERSITY of NORTH CAROLINA at CHAPEL HILL

Doctoral Written Examination in Physics, 2015

Part I: Classical Mechanics

Friday, May 8, 2015

Instructions: Please work in the assigned room, but take a break outside any time you wish. Mathematical handbooks and electronic calculators are allowed.

Begin each answer on a new sheet and write only on one side of each sheet.
Identify each sheet by:

Page Number_____ of Question_____ and your Identification Number______

CM: Classical Mechanics
Work 3 out of 5 problems

(Partial credit will be given for partial answers)

My work is completed in full observance of the Honor code:

Signature_________________________

Print name_______________________

UNIVERSITY of NORTH CAROLINA at CHAPEL HILL

Doctoral Written Examination in Physics, 2015

Part I: Statistical Mechanics

Friday, May 8, 2015

Instructions: Please work in the assigned room, but take a break outside any time you wish. Mathematical handbooks and electronic calculators are allowed.

Begin each answer on a new sheet and write only on one side of each sheet.
Identify each sheet by:

Page Number_____ of Question_____ and your Identification Number______

SM: Statistical Mechanics
Work 3 out of 5 problems

(Partial credit will be given for partial answers)

My work is completed in full observance of the Honor code:

Signature_________________________

Print name_______________________

UNIVERSITY of NORTH CAROLINA at CHAPEL HILL

Doctoral Written Examination in Physics, 2015

Part II: Electromagnetism I

Monday, May 11, 2015

Instructions: Please work in the assigned room, but take a break outside any time you wish. Mathematical handbooks and electronic calculators are allowed.

Begin each answer on a new sheet and write only on one side of each sheet.
Identify each sheet by:

Page Number_____ of Question_____ and your Identification Number______

EMI: Electromagnetism I
Work 3 out of 5 problems

(Partial credit will be given for partial answers)

My work is completed in full observance of the Honor code:

Signature_________________________

Print name____________________________

UNIVERSITY of NORTH CAROLINA at CHAPEL HILL

Doctoral Written Examination in Physics, 2015

Part II: Quantum Mechanics I

Monday, May 11, 2015

Instructions: Please work in the assigned room, but take a break outside any time you wish. Mathematical handbooks and electronic calculators are allowed.

Begin each answer on a new sheet and write only on one side of each sheet.
Identify each sheet by:

Page Number_____ of Question_____ and your Identification Number______

QMI: Quantum Mechanics I
Work 3 out of 5 problems

(Partial credit will be given for partial answers)

My work is completed in full observance of the Honor code:

Signature_________________________

Print name____________________________

UNIVERSITY of NORTH CAROLINA at CHAPEL HILL

Doctoral Written Examination in Physics, 2015

Part III: Electromagnetism II

Monday, May 11, 2015

Instructions: Please work in the assigned room, but take a break outside any time you wish. Mathematical handbooks and electronic calculators are allowed.

Begin each answer on a new sheet and write only on one side of each sheet.
Identify each sheet by:

Page Number_____ of Question_____ and your Identification Number______

EMII: Electromagnetism II
Work 3 out of 5 problems

(Partial credit will be given for partial answers)

My work is completed in full observance of the Honor code:

Signature_________________________

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UNIVERSITY of NORTH CAROLINA at CHAPEL HILL

Doctoral Written Examination in Physics, 2015

Part III: Quantum Mechanics II

Monday, May 11, 2015

Instructions: Please work in the assigned room, but take a break outside any time you wish. Mathematical handbooks and electronic calculators are allowed.

Begin each answer on a new sheet and write only on one side of each sheet.
Identify each sheet by:

Page Number_____ of Question_____ and your Identification Number______

QMII: Quantum mechanics II
Work 3 out of 5 problems

(Partial credit will be given for partial answers)

My work is completed in full observance of the Honor code:

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UNIVERSITY of NORTH CAROLINA at CHAPEL HILL

Doctoral Written Examination in Physics, 2015

Part III: Astro I (Stellar)

Monday, May 11, 2015

Instructions: Please work in the assigned room, but take a break outside any time you wish. Mathematical handbooks and electronic calculators are allowed.

Begin each answer on a new sheet and write only on one side of each sheet.
Identify each sheet by:

Page Number_____ of Question_____ and your Identification Number______

Astro I (Stellar):
Work 3 out of 5 problems

(Partial credit will be given for partial answers)

My work is completed in full observance of the Honor code:

Signature_________________________

Print name____________________________

UNIVERSITY of NORTH CAROLINA at CHAPEL HILL

Doctoral Written Examination in Physics, 2015

Part III: Astro II (Cosmology)

Monday, May 11, 2015

Instructions: Please work in the assigned room, but take a break outside any time you wish. Mathematical handbooks and electronic calculators are allowed.

Begin each answer on a new sheet and write only on one side of each sheet.
Identify each sheet by:

Page Number_____ of Question_____ and your Identification Number______

Astro II (Cosmology):
Work 3 out of 5 problems

(Partial credit will be given for partial answers)

My work is completed in full observance of the Honor code:

Signature_________________________

Print name____________________________

SM-1

Consider an ideal gas in a one-dimensional channel of length L. The energy of the particle is given by $E = (p^2/2m) - \epsilon_0$.

(a) Show, using the classical approach, that the partition function of one particle is given by

$$Q_1(T, L) = \frac{L}{\lambda} \exp(\epsilon_0/kT) \quad \text{and} \quad \lambda = \frac{h}{\sqrt{2\pi mkT}}$$

(b) Calculate the chemical potential of this system of N indistinguishable particles at temperature T.

Reminder: $\int_0^\infty e^{-x^2} dx = \sqrt{\pi}/2$.

SM-2

Consider a white dwarf star where the number of electrons is N, the mass of the star is $M = 2Nm_p$ where m_p is the mass of the proton, and the volume of the star is V. The pressure of an ideal Fermi gas is given by

$$P = 2\frac{4\pi}{3h^3} \int_0^\infty \frac{1}{e^{(\epsilon-\mu)/kT} + 1} \left(p\frac{\partial \epsilon}{\partial p} \right) p^2 dp,$$

where ϵ is the relativistic kinetic energy given by

$$\epsilon = m_e c^2 \left\{ \left[1 + (p/m_e c)^2 \right]^{1/2} - 1 \right\}.$$

It can be shown that the Fermi momentum is given by $p_F = (\frac{3N}{8\pi V})^{1/3} h$, where h is the Planck constant.

Show that in the $T = 0$ limit, the radius of the star R is given by the equation

$$\frac{\pi m_e^4 c^5}{3h^3} 8 \int_0^{\theta_F} \sinh^4 \theta \, d\theta = \frac{\alpha}{4\pi} \frac{GM^2}{R^4}, \qquad \text{where} \qquad m_e c \sinh \theta_F = p_F.$$

Here m_e is the mass of the electron, c is the speed of light, $\alpha \simeq 1$ is a known constant, and G is the gravitational constant.

SM-3

Consider the Ising model

$$H\{\sigma_i\} = -J \sum_{nn} \sigma_i \sigma_j - \mu B \sum_{i=1}^{N} \sigma_i$$

where J is a constant representing the exchange coupling, nn means the nearest neighbors, μ is the magnetic moment, B is the magnetic field, and σ_i assumes either the number +1 for an "up" spin or -1 for a "down" spin. In the context of the Bragg-Williams approximation, show that the critical temperature T_c of the spontaneous magnetization is given by $T_c = qJ/k$, where k is the Boltzmann constant and q is the number of the nearest neighbors.

SM-4

We are interested in some basic properties of the density matrix in quantum statistical mechanics. Consider a system with Hamiltonian H. Let the set of normalized states $|\psi_k(t)\rangle$ be an ensemble of possible states of the system obeying the Schrodinger equation. The density matrix is given by $\rho = \sum_k p_k |\psi_k\rangle \langle\psi_k|$.

(a) Given that the trace of ρ^2 is equal to 1 for pure state and less than 1 for mixture, show that a pure state cannot evolve into a mixture or vice versa by considering $\frac{\partial \mathrm{Tr}\rho^2}{\partial t}$.

(b) Show that equilibrium statistical mechanics is described by a density matrix of the form $\rho = \rho(H)$ by considering $\frac{\partial \rho}{\partial t}$.

SM-5

Find the distribution function (the average occupation number of a given state with an energy ϵ at a temperature T) for the species which can occupy that quantum state in arbitrary numbers, ranging from 0 up to N. Show that in the cases of $N = 1$ and $N = \infty$ one recovers the Dirac and Bose statistics, respectively.

CM-1

A small object of mass m slides without friction on the inner side of the surface of rotation described by the equation $z = \alpha r^4$ in the three-dimensional parabolic coordinates (r, θ, z).

(a) Write down the Lagrangian of the system (don't forget gravity), complete with the Lagrange multiplier λ implementing the above constraint, and derive the Euler-Lagrange equations (2 pts).

(b) Consider a circular (horizontally oriented) orbit with angular momentum l. Find its radius r_0 (1 pt), angular velocity $\dot{\theta}$ (1 pt), and the normal reaction force N (2 pts) which is related (but is not equal) to λ.

(c) For the above orbit, compute the ratio between the kinetic K and potential U energies, and compare it to the prediction of the virial theorem (which may or may not be applicable, since the overall three-dimensional force acting on the object is not central) (2 pts).

(d) Consider small radial oscillations about the above circular orbit, expand the Lagrange equations to first order in δr, and determine the frequency ω_r of such oscillations (2 pts).

CM-2

A particle of mass m is moving in the two-dimensional rotationally invariant potential $U(r) = \alpha r^6$ (assume no gravity).

(a) Using the radial r and angular θ coordinates, together with their conjugate momenta p_r and p_θ, write down the Hamiltonian (1 pt) and derive the Hamilton equations (1 pt).

(b) Derive the Hamilton-Jacobi equation in terms of the principal function $S(r, \theta; E, l; t)$ (1 pt).

(c) Use separation of variables to find the function S (2 pts) and derive the equation of a general orbit $r(\theta)$ in the closed integral form (but don't do the integral) (2 pt).

(d) Consider the orbit with $l = 0$ and allow for an adiabatic change in the potential's strength $\alpha(t)$. Find the exponent η in the resulting time dependence of the energy $E(t) \sim \alpha(t)^\eta$ upon $\alpha(t)$ by computing the adiabatic invariant $J_r = \oint p_r dr$ (3 pts).

CM-3

Consider the Hamiltonian $H = \frac{1}{2I}l^2 + \alpha \sin\theta$ where $\theta(t)$, $l(t)$, and I are the angular variable, conjugate angular momentum, and moment of inertia, respectively.

(a) In the unperturbed case of $\alpha = 0$, identify the bare action-angle variables $w_0(t) = \nu_0 t$ and $J_0 = \oint l d\theta$, obeying the Poisson bracket $[w_0, J_0] = 1$ (2 pts), as well as the corresponding energy $E_0(J_0)$ and frequency $\nu_0 = dE_0/dJ_0$ (2 pts).

(b) Treat the term proportional to α as a perturbation and use time-independent perturbation theory to compute 1^{st} order corrections to the energy (δE) (1 pt) and frequency ($\delta\nu$)(1 pt).

(c) Find, to 1^{st} order in α, the type-II generating function $S(w_0, J) = w_0 J + S_1(w_0, J)$ which performs a canonical transformation from $w_0(t)$, $J_0(t)$ to the exact action-angle variables $w(t) = \nu t$ and $J(t) = const$ (2 pts).

(d) Use the above results to express $\theta(t)$ and $l(t)$ in terms of the exact action-angle variables with their known time dependencies ($w(t) = \nu t$ and $J(t) = const$), thus obtaining the solution of the equations of motion to 1^{st} order in α. (2 pts).

CM-4

A bead of mass M is able to move without friction along a stationary horizontal rod (directed along the x axis). In addition, a second body of mass m is attached to the first bead and suspended below it via a massless rod of length a. This second mass and rod form a pendulum that is able to swing in the xy-plane (where y is the vertical axis).

(a) Obtain the Lagrangian for the system of two masses.

(b) Assuming small amplitude oscillations, determine the frequency at which the pendulum oscillates.

CM-5

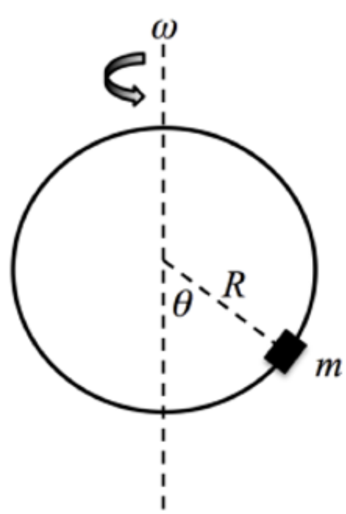
A bead of mass m is constrained to slide without friction on a circular hoop of radius R. The hoop is oriented vertically and is attached to a motor that rotates it at a constant angular speed ω, as shown in the attached figure. The bead experiences a constant gravitational force directed downward, given as mg. Answer the following questions:

(a) Find the Lagrangian for this system using appropriate variables.

(b) Find the effective potential.

(c) Find the equilibrium points for this system. For each point, state if the point is stable or unstable. Also state what, if any, requirements there are on the given quantities $(m; R; g; \omega)$ for these points to exist.

(d) At least one of the points you found earlier is stable. Find the period of small oscillations around the equilibrium point for one of the stable points. If there is more than one, pick the one that is easiest to compute.



EM-I-1

A two-dimensional semi-infinite, empty slot is located between $x = 0$ and $x = a$, and $y > 0$. The $x = 0$ and $x = a$ sides are grounded (i.e., held at vanishing electric potential), and the side at $y = 0$ is held at a constant potential ϕ_0. Find the potential inside the slot and determine its asymptotic behavior when $y \gg a$.

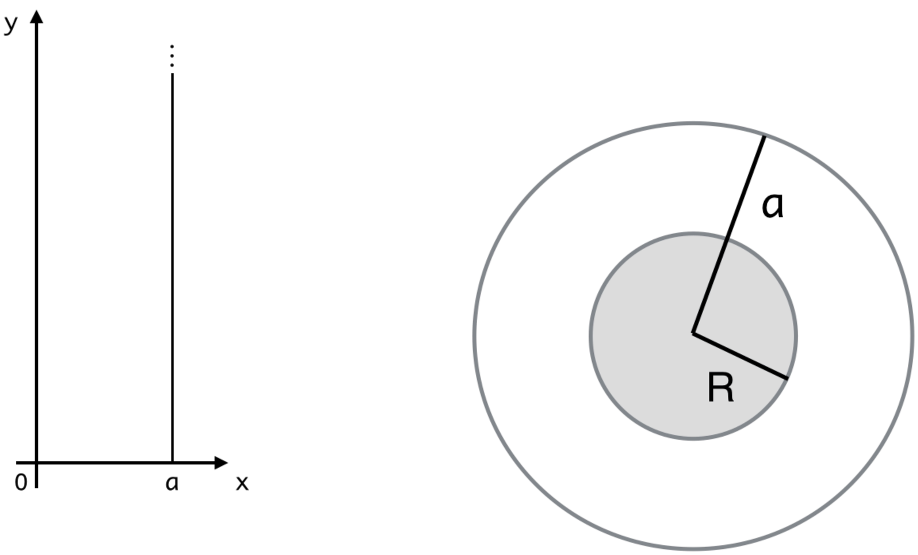


Figure 1: Geometry for problem 1 (left). Geometry for problem 2 (right).

EM-I-2

A set of known constants α_n parameterizes an external potential in an empty spherical volume $r < a$ as

$$\phi_{\text{ext}}(r, \theta) = \sum_{n=1}^{\infty} \alpha_n \left(\frac{r}{R}\right)^n P_n(\cos\theta).$$

Here R ($R < a$) is the radius of a smaller solid, grounded conducting sphere that is also centered on the origin.

(a) Find the total electrostatic potential due to the conducting sphere and the external field.

(b) Find the induced charge density σ on the surface of the inner conducting sphere.

EM-I-3

A conducting spherical shell of radius a is placed inside another conducting spherical shell of radius $b > a$. The center of the inner shell is displaced a distance $c \ll a, b$ from the center of the outer shell. The inner shell has a given charge Q, while the outer one is grounded.

(a) Show that, to first order in c/b, the equation describing the outer sphere, from the center of the inner sphere, is given by

$$r(\theta, \phi) = b\left(1 + \frac{c}{b}\cos\theta\right),$$

where θ is the angle between the radius r and the line segment joining the two centers.

(b) Assuming that the potential $\Phi(r, \theta)$ in the region between the shells contains only $\ell = 0$ and $\ell = 1$ angular components, show that the total charge on the inner shell determines one of the unknown coefficients.

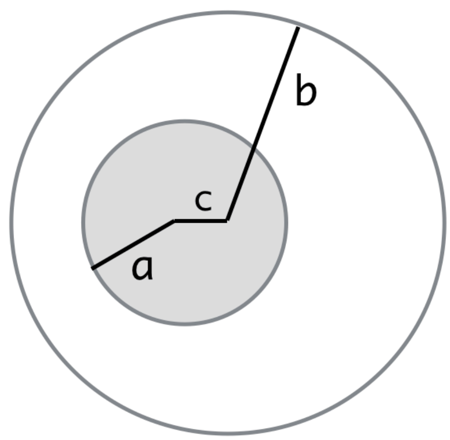


Figure 2: Geometry for problem 3.

EM-I-4

A thin disk of radius a carries a uniform surface charge density σ and is centered in the x, y plane. The disk is rotating about the z axis with angular velocity ω.

(a) Calculate the direction and magnitude of the (non-zero) leading order part of the electric field far from the disk at point (x, y, z).

(b) Calculate the non-zero leading order part of the magnetic field far from the disk at point (x, y, z).

EM-I-5

Suppose that the electrostatic potential $\phi(x, y, z)$ in empty space were governed by the equation

$$\nabla^2\phi - m\phi = 0,$$

where m is a positive constant.

(a) Find the general solution to this equation in rectangular coordinates.

(b) For a given choice of boundary conditions, for example, on a large sphere, suppose that there are two solutions in the interior, ϕ_1 and ϕ_2, that satisfy the same boundary condition. Show that, in fact, these solutions are the same; that, in other words, any solution is unique. For this, you may want to use Green's First Identity:

$$\int_V \Psi\nabla^2\Phi \, dv = -\int_V \nabla\Psi \cdot \nabla\Phi \, dv + \oint_S \Psi\nabla\Phi \cdot d\mathbf{S}.$$

What now is true if $m < 0$?

QM-I-1

A nucleus of total angular momentum $j = 1$ interacts with an electron in an s-wave state. The important states are ones with well-defined two-particle angular momentum J.

(a) What values of J are allowed?

(b) Find the Clebsch-Gordan coefficient (in Sakurai notation) $\langle 1\tfrac{1}{2}; 1\tfrac{1}{2}|1\tfrac{1}{2}; \tfrac{3}{2}\tfrac{3}{2}\rangle$.

(c) Find $\langle 1\tfrac{1}{2}; 0\tfrac{1}{2}|1\tfrac{1}{2}; \tfrac{3}{2}\tfrac{1}{2}\rangle$ and $\langle 1\tfrac{1}{2}; 1\text{-}\tfrac{1}{2}|1\tfrac{1}{2}; \tfrac{3}{2}, \tfrac{1}{2}\rangle$. It might help to know that

$$J_-|j, m\rangle = \hbar\sqrt{(j+m)(j-m+1)}|j, m-1\rangle\,.$$

(d) Find $\langle 1\tfrac{1}{2}; 1\tfrac{1}{2}|0\tfrac{1}{2}; \tfrac{1}{2}\tfrac{1}{2}\rangle$ and $\langle 1\tfrac{1}{2}; 1\text{-}\tfrac{1}{2}|1\tfrac{1}{2}; \tfrac{1}{2}\tfrac{1}{2}\rangle$

(e) Find the other 4 nonzero Clebsch-Gordan coefficients. It might help to know that

$$\langle j_1 j_2; m_1 m_2|j_1 j_2; JM\rangle = (-1)^{j_1+j_2-J}\langle j_1 j_2; -m_1 - m_2|j_1 j_2; J - M\rangle\,.$$

QM-I-2

Consider a system with a pair of observable quantities A and B, whose commutation relations with the Hamiltonian take the form:

$$[H, A] = igB\,, \quad [H, B] = -igA\,,$$

where g is some real constant.

(a) Write down the Heisenberg equations of motion for A and B.

(b) Suppose the expectation values of A and B at time zero are $\langle A\rangle_0$ and $\langle B\rangle_0$. Find the expectation values $\langle A\rangle$ and $\langle B\rangle$ at later times t.

QM-I-3

A particle of mass m is trapped in an infinite one-dimensional potential well of width L.

(a) The particle is in the ground state of the potential when the width of the well is suddenly increased to $2L$. What is the probability that the particle will be in the ground state of the expanded well? (4 points)

(b) Assume that the particle is in the ground state of the expanded well. The walls of the well are suddenly dissolved and the particle becomes free. Find the momentum probability distribution of the particle after it is freed. (6 points)

You may find the following indefinite integral useful for this problem:

$$\int \sin{(ax)} \sin{(bx)} dx = \frac{\sin{(a-b)x}}{2(a-b)} - \frac{\sin{(a+b)x}}{2(a+b)}$$

QM-I-4

An element of the rotation group can be written as the matrix $D(R(\vec{e}, \theta)) = e^{-i\vec{e}\cdot\vec{J}\theta}$, where $\vec{J}$ are the three angular momentum generators, and $\vec{e}, \theta$ are the axis and angle of the rotation.

Give the elements of the rotation group for a rotation of angle θ around the z-axis that describe

(a) a spin 0 particle

(b) a spin $\frac{1}{2}$ particle.

(c) the combined system of two spin $\frac{1}{2}$ particles, and prove explicitly they satisfy the rule for how a direct product representation decomposes into irreducible representations.

QM-I-5

Compute the eigenstates and eigenvalues of the one-dimensional harmonic oscillator with Hamiltonian $H = \frac{p^2}{2m} + \frac{kx^2}{2}$. Transform to creation and annihilation operators, in which $[a, a^\dagger] = 1$. Find the states of the system in terms of the operators acting on the vacuum state. Give the energy eigenvalues of these states, and label the values of any parameters.

EM-II-1

Consider a cold collisional plasma with a scalar permittivity given by

$$\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega(\omega + i\nu)},$$

where ν is the electron-ion collision rate, ω is the frequency of oscillation, and ω_p is the plasma frequency.

(a) Write down the Fourier transformed forms of Faraday's law and Ampere's law (in terms of ω and $\vec{k}$). Assume all of the effects of permittivity are swept into $\epsilon(\omega)$ such that $D_j = \epsilon(\omega)\delta_{jk}E_k$.

(b) Obtain the dispersion relation for electrostatic waves (i.e., longitudinal waves) in the collisional plasma.

(c) If an initial spatial longitudinal wave existed in the plasma with real wavevector $\vec{k}$, what would be the time dependence of this mode and what is the primary effect of the collision rate ν?

EM-II-2

A parallel-plate capacitor is made of circular plates of radius a separated by a distance d. The voltage across the plates has the time dependence $V = V_0 \cos(\omega t)$ (supplied by long, narrow, zero-resistance wires). Assume that $d \ll a \ll c/\omega$, so that fringe and retardation effects can be neglected. (In other words, everything is instantaneous and we are interested in the fields deep inside the capacitor, near the symmetry axis.)

(a) Use Maxwell's equations and symmetry arguments to determine the electric and magnetic fields in the region in between the plates as functions of time.

(b) Find the time-varying charge on the plates.

(c) Using Ampere's law, find the magnetic field in the region outside the capacitor (think of the previous region, but on the other side of each plate). Relate the discontinuity in the magnetic field across the plates to the current density within the plates.

EM-II-3

Suppose that in an electron-positron collider the following event is observed

$$e^+ + e^- \to \pi^0 + \pi^0$$

with the electron and positron having equal energy E_e and opposite momentum. The electron's mass is denoted by m_e and pion's mass is labeled as m_π. The pions of course emerge from the event with opposite momentum, equal energy, and some Lorentz factor γ_π.

The two pions each subsequently decay into a pair of gamma rays

$$\pi^0 \to \gamma_1 + \gamma_2,$$

which is the most common decay mode. In the laboratory frame the photon energies are E_1 and E_2.

(a) Expressed in terms of the various parameters (i.e., combinations of some or all of E_e, m_e, m_π), what is the minimum possible photon energy E_1 and the maximum possible photon energy E_2 measured in the laboratory frame?

(b) Given that $m_\pi = 135$ MeV/c^2, if the electron and positron both have energy of $E_e = 351$ MeV/c^2, what are the numerical values of E_1 and E_2?

EM-II-4

The goal of this problem is to prove that transverse electromagnetic (TEM) waves cannot occur in a hollow wave guide composed of a perfect conductor. Consider a monochromatic wave propagating in a hollow pipe of arbitrary but uniform (in x) cross sectional shape, with the electric field of the form

$$\vec{E}(x, y, z, t) = \vec{E}_0(y, z)e^{i(kx-\omega t)},$$

and similar form for $\vec{B}(x, y, z, t)$. Assume that a TEM wave exists ($E^x = B^x = 0$).

(a) What is the boundary condition for $\vec{E}$ at the inner wall of the wave guide?

(b) Use Gauss's law and Faraday's law to show that $\vec{E}_0$ has zero divergence and zero curl.

(c) Show that the corresponding scalar potential satisfies Laplace's equation.

(d) Use (a) and (c) to show that $\vec{E}_0$ is zero, and hence no wave at all.

EM-II-5

The general expression for the spectral and angular distribution of energy radiated from a relativistic electron is

$$\frac{dW}{d\omega d\Omega} = \frac{e^2\omega^2}{4\pi^2 c}\left|\int dt'\,\vec{n}\times(\vec{n}\times\vec{\beta}(t'))\,\exp[i\omega(t'-\vec{n}\cdot\vec{x}(t')/c)]\right|^2 .$$

Let the observation direction be $\vec{n}$. Adopt as well two real orthonormal vectors $\vec{e}^A$ and $\vec{e}^B$ that are also orthogonal to $\vec{n}$. These three vectors make up an orthonormal triad and with them the three-dimensional identity can be decomposed

$$\delta_{ij} = n_i n_j + e_i^A e_j^A + e_i^B e_j^B .$$

This decomposition is useful for handling the vector $\vec{n}\times(\vec{n}\times\vec{\beta}(t'))$ (hint: consider $\beta_i = \delta_{ij}\beta_j$). Assume that a relativistic electron undergoes an abrupt collision with a heavy (fixed) nucleus and experiences a change in its velocity from $\vec{\beta}'$ to $\vec{\beta}''$ (both constant). At low frequencies ω the Bremsstrahlung spectrum can be treated like transition radiation.

(a) Show that the spectrum has the form

$$\frac{dW}{d\omega d\Omega} = \frac{e^2}{4\pi^2 c}\sum_{m=A,B}\left|\vec{e}^m\cdot\left(\frac{C\vec{\beta}''}{1-\vec{n}\cdot\vec{\beta}''}+\frac{D\vec{\beta}'}{1-\vec{n}\cdot\vec{\beta}'}\right)\right|^2 ,$$

and find the unknown coefficients C and D.

(b) What role do the unit vectors $\vec{e}^A$ and $\vec{e}^B$ play?

(c) Explain clearly what is being coherently summed and what is being incoherently summed.

QM-II-1

Two identical spin 1/2 fermions move in one dimension under the influence of an infinite-wall potential $V = \infty$ for $x < 0, x > L$, and $V = 0$ for $0 \leq x \leq L$.

(a) Write the ground state wave function and the ground state energy when the two particles are constrained to a triplet spin state (ortho state).

(b) Repeat (a) when they are in a singlet spin state (para state).

(c) Let us now suppose that the two particles interact mutually via very short range attractive potential that can be approximated by $V = -\lambda\delta(x_1 - x_2), \lambda > 0$. Assuming that perturbation theory is valid even with such a singular potential, discuss semi-quantitatively what happens to the energy levels obtained in $(a), (b)$.

QM-II-2

Show that the slow turn on of perturbation $V = V_0 e^{\eta t}$ can generate contributions in the transition probability from the second order term. Discuss the reason and compare the strength of first order to second order corrections.

QM-II-3

Consider a model of the proton in which it is a spherical particle that has a uniform *surface* charge distribution and a radius R.

(a) Write down the potential of the proton in this model as a function of the radius from its center (r) and its total charge (q). (2 points)

(b) Write down the effect of this model as a perturbation to the original Hydrogen Hamiltonian, which assumes the proton is a point-like particle. (2 points)

(c) Find the correction in the ground state energy of hydrogen due to the finite size of the proton in this model. You may assume that $R \ll a_0$ where a_0 is the Bohr radius. You may express your answer in terms of the Bohr radius (6 points).

QM-II-4

A sample of spin-1/2 particles is placed in a constant magnetic field and a much weaker perpendicular oscillating magnetic field. Each spin is governed by the Hamiltonian

$$H = \hbar \left[\frac{b_0}{2} \sigma_z + b_1 \sigma_x \cos(\omega t) \right] ,$$

where ω, b_0, and b_1 are positive constants with $b_1 \ll b_0$.

(a) Suppose at $t = 0$ a particle is in the z-spin "down" state. Use first-order perturbation theory to find the probability that after time t the particle is found in the z-spin "up" state.

(b) Suppose $\hbar\omega$ is very close $\hbar b_0$, the unperturbed energy difference between the up and down states. Keep the largest terms to simplify the answer you obtained above.

(c) For how long, roughly, does the probability increase without oscillating? Do you obtain a constant transition *rate* during this time? If not, how do you reconcile your result with the Fermi's golden rule?

QM-II-5

Two identical spin-$\frac{1}{2}$ particles are trapped in the same infinite potential well of width L. They experience a contact spin-interaction as follows:

$$H' = A\,\delta(x_1 - x_2)\,\mathbf{S_1} \cdot \mathbf{S_2}$$

where A is a constant, x_i are the positions of the two particles, and $\mathbf{S_i}$ are their spin vectors.

(a) Assume $A = 0$ and write down all the wave functions and energies for the ground and first excited states. (5 points)

(b) Assume now that $A > 0$. Find the correction to the ground state energy using first order perturbation theory. (5 points)

You may find the following indefinite integral useful for this problem:

$$\int \sin^4(x) dx = \frac{1}{32} \sin(4x) - \frac{1}{4} \sin(2x) + \frac{3}{8} x$$

AS-I-1

A star has uniform opacity κ, uniform $\beta \equiv P_{\text{gas}}/P$, and a convective core outside of which there is no thermonuclear energy generation.

(a) Show that the mass fraction of the core is

$$\frac{M_c}{M} = \frac{\gamma_{\text{ad}}}{4(\gamma_{\text{ad}} - 1)(1-\beta)} \frac{\kappa L}{4\pi c G M}$$

with γ_{ad} the adiabatic value.

AS-I-2

Completely black (absorbing) dust grains of average radius 1 micron are heated to 3000 K by an active galactic nucleus of central (dark) mass 10^7 $M_\odot$. The heat source is the inner accretion disk of radial extent $\sim 0 - 1$ AU from which we detect soft X-rays with spectrum approximately a 10^6 K black body. The emitting dust grains see the entire ring at optical depth $\tau = 15$ around peak emission wavelength.

(a) Show that the ring emits 4×10^{39} W.

(b) Show that at 1 pc distance each grain absorbs 3.2×10^{-13} W.

(c) Show that a heated grain emits 2.1×10^{-15} W.

(d) Show that the average orbital radius of the dust grain is ~ 12 pc.

(e) Assuming 100$\times$ as much mass within the grain's orbit than in the central mass, and that the grain sublimates on average in 1000 years, how many orbits does the grain make before sublimating?

AS-I-3

From writing the stellar structure equations in Lagrangian form, homology shows that $L \propto M^3$ and $R_* \propto M^{\frac{n-1}{n+3}}$ for energy generation $dF = q_o \rho T^n dM$.

(a) Use solar values to calculate T_{eff} in K at the lowest mass end $M = 0.1 M_\odot$ of the stellar main sequence where ${}^1H$ thermonuclear fusion can still occur. Use the appropriate value of n for the hydrogen fusion rate.

(b) Assuming solar composition, what is the approximate main sequence lifetime of this star, what is its lifetime on the horizontal branch, and what remnant does it leave?

AS-I-4

The UV spectrum from an accretion disk around a Galactic micro-quasar shows the He^+ Lyman edge at 22.8 nm wavelength across which flux jumps $4\times$ from shorter to longer wavelengths. Assume LTE and that the disk is pure helium. Ignore stimulated emission, and assume that opacity is due only to bound-free absorption.

(a) Considering the relevant bound-free opacities that scale $\propto \lambda^{-3} n^{-5}$ for level n, show that this part of the accretion disk has $T \sim 64,000$ K.

(b) A model predicts that this region is $\sim 1/4$ AU from the micro-quasar. What is the maximum mass of the micro-quasar that still allows this gas to escape from the system (moving at up to three times rms gas velocity)? Assume no wind and negligible mass in the accretion disk.

AS-I-5

A spherical cloud of radius R and uniform density ρ lies a distance D from an observer. The cloud subtends angle θ_o on sky. The attenuation coefficient κ is constant both within the cloud and with frequency. Assume that all rays through the cloud are parallel.

(a) Show that the optical depth through the cloud as function of $\theta < \theta_o$ is given by

$$\tau(\theta) = 2l(\theta)\kappa\rho = \tau_o\sqrt{1 - (\tan\theta / \tan\theta_o)^2}$$

(b) Show that the angular radius of the optically thick cloud, θ_1, is given by

$$\theta_1 = \theta_o\sqrt{1 - (2\kappa\rho R)^{-2}}$$

Potentially Useful Constants

Name	Symbol	Value	Unit
Gravitational constant	G, κ	$6.67259 \cdot 10^{-11}$	$m^3kg^{-1}s^{-2}$
Speed of light in vacuum	c	$2.99792458 \cdot 10^{8}$	m/s (def)
Planck's constant	h	$6.6260755 \cdot 10^{-34}$	Js
Dirac's constant	$\hbar = h/2\pi$	$1.0545727 \cdot 10^{-34}$	Js
Rydberg's constant	Ry	13.595	eV
Reduced mass of the H-atom	μ_{H}	$9.1045755 \cdot 10^{-31}$	kg
Stefan-Boltzmann's constant	σ	$5.67032 \cdot 10^{-8}$	$Wm^{-2}K^{-4}$
Boltzmann's constant	k_B	$1.3806 \cdot 10^{-23}$	$J\ K^{-1}$
Wien's constant	k_{W}	$2.8978 \cdot 10^{-3}$	mK
Molar gasconstant	R	8.31441	$J \cdot mol^{-1} \cdot K^{-1}$
Avogadro's constant	N_{A}	$6.0221367 \cdot 10^{23}$	mol^{-1}
Boltzmann's constant	$k = R/N_{\mathrm{A}}$	$1.380658 \cdot 10^{-23}$	J/K
Electron mass	m_{e}	$9.1093897 \cdot 10^{-31}$	kg
Proton mass	m_{p}	$1.6726231 \cdot 10^{-27}$	kg
Neutron mass	m_{n}	$1.674954 \cdot 10^{-27}$	kg
Elementary mass unit	$m_{\mathrm{u}} = \frac{1}{12} m({}^{12}_{6}\mathrm{C})$	$1.6605656 \cdot 10^{-27}$	kg
Diameter of the Sun	$D_{\odot}$	$1392 \cdot 10^{6}$	m
Mass of the Sun	$M_{\odot}$	$1.989 \cdot 10^{30}$	kg
Radius of Earth	R_{A}	$6.378 \cdot 10^{6}$	m
Mass of Earth	M_{A}	$5.976 \cdot 10^{24}$	kg
Earth orbital period	Tropical year	365.24219879	days
Astronomical unit	AU	$1.4959787066 \cdot 10^{11}$	m
Light year	lj	$9.4605 \cdot 10^{15}$	m
Parsec	pc	$3.0857 \cdot 10^{16}$	m

AS-II-1

(a) Starting from the first Friedmann equation and the First Law of Thermodynamics, derive an expression for the Hubble parameter $H(a)$ for a universe containing matter, radiation, and an additional substance with an equation of state $p = w\rho$, where $w < -1/3$ and w is a slowly varying function of time, which for the purpose of integration you can take to be constant. Your final expression should only depend on H_0, w, the present-day ratios of the different densities to the critical density, and a. **Do not assume that the Universe is flat.** (4 pts.)

(b) For the universe described above, derive expressions for the scale factor $a(t)$ in the limit of large a for $-1/3 > w > -1$, $w = -1$ and $w < -1$. In a few sentences, discuss the ultimate fate of this universe in these three scenarios. (6 pts.)

AS-II-2

The tensor-to-scalar ratio r is related to the inflationary potential $V(\phi)$ via the slow-roll parameter ϵ_V: $r \simeq 16\epsilon_V$, where

$$\epsilon_V = \frac{m_{\rm Pl}^2}{16\pi}\left[\frac{V'(\phi)}{V(\phi)}\right]^2$$

is evaluated when the perturbation mode exits the horizon during inflation. The Planck mass is $m_{\rm Pl} = 1.221 \times 10^{19}\,\text{GeV} = \sqrt{\frac{\hbar c}{G}}$.

Consider slow-roll inflation with a quadratic inflationary potential: $V(\phi) = (1/2)m^2\phi^2$.

(a) What is the value of ϕ when inflation ends? (2 pts.)

(b) If inflation ends when the scale factor is a_e, derive the following expression for ϕ as a function of a during inflation (4 pts):

$$\phi = \sqrt{\frac{m_{\rm Pl}^2}{2\pi}\left(\ln\left[\frac{a_e}{a}\right] + \frac{1}{2}\right)}.$$

(c) Assume that the Universe became radiation-dominated immediately after inflation ended and that the reheat temperature was 10^{14} GeV. What is r for a perturbation mode with $k = 0.002\ \text{Mpc}^{-1}$? (4 pts.)

AS-II-3

Consider the formation of atomic hydrogen in the early Universe:

$$p^+ + e^- \longleftrightarrow H^0 + \gamma.$$

The binding energy of atomic hydrogen is $B = 13.6$ eV, and $g_H = 4$, $g_p = 2$, and $g_e = 2$. The ionization fraction X_e is defined as the number density of free electrons (n_e) divided by the total number density of electrons ($n_e + n_H$).

(a) Derive the Saha equation for the ratio

$$\frac{X_e^2}{1 - X_e}$$

in terms of the mass of the electron, the temperature, the binding energy of hydrogen, and the baryon-to-photon ratio ($\eta \equiv n_b/n_\gamma$). (7 pts.)

(b) The binding energy of 13.6 eV is equivalent to a temperature of about 1.5×10^5 K. Yet, in the early universe, recombination occurred ($X_e < 0.1$) only after the temperature dropped to $T \simeq 3500$ K. What cosmological parameter is responsible for this delay? Show this fact mathematically using the Saha equation and provide a physical explanation. (3 pts.)

Useful information: In natural units, the number density of a nonrelativistic particle species that is in thermodynamic equilibrium at a temperature T is

$$n_i = g_i \left(\frac{m_i T}{2\pi} \right)^{3/2} e^{-m_i/T},$$

where g_i is the internal degrees of freedom and m_i is the mass of the particle. The number density of photons is

$$n_\gamma = 2 \frac{1.202}{\pi^2} T^3.$$

AS-II-4

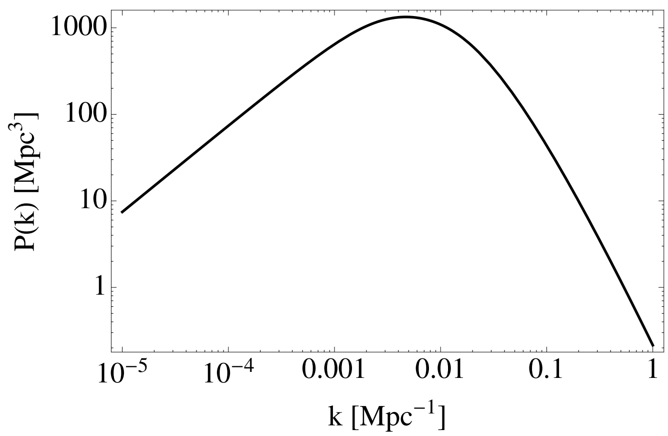


Figure 1: Matter power spectrum for use in problem 4.

This figure shows a matter power spectrum $P_\delta(k)$ in a universe with $\Omega_r h^2 = 4.15 \times 10^{-5}$.

(a) Estimate $\Omega_m h^2$ for the universe that has this matter power spectrum. (5 pts)

(b) Assume that this universe has the same Hubble constant ($H_0 = 68$ km/s/Mpc), dark energy density ($\Omega_{\rm de}$=0.7), and primordial curvature power spectrum $\mathcal{P}_\zeta(k)$ as our universe. Would structure form earlier or later in this universe as compared to our universe? Why? (2 pts.)

(c) Again, assume that this universe has the same H_0, $\Omega_{\rm de}$, and $\mathcal{P}_\zeta(k)$ as our universe. Qualitatively describe how the anisotropy spectrum of the cosmic microwave background in this Universe would differ from the anisotropy spectrum in our Universe. At least two distinct changes in the anisotropy spectrum must be identified for full credit, and sketches are highly encouraged. (3 pts.)

AS-II-5

Consider a flat universe with $\Omega_m = 0.3$, $\Omega_{\rm de} = 0.7$, and $\Omega_r = 9.0 \times 10^{-5}$. Assume that dark energy has an equation of state parameter $w = -1$, and take the Hubble constant to be $H_0 = 68$ km/s/Mpc.

(a) What is the redshift of matter-radiation equality? What is the redshift of dark energy-matter equality? (2 pts.)

(b) Estimate the angular size of the particle horizon at a redshift of 1090 by keeping only the dominant energy component during radiation domination, matter domination, and dark energy domination, respectively. (6 pts.)

(c) Discuss how your result in part (b) provides motivation for inflation. (2 pts.)

Potentially Useful Constants

$$
\begin{aligned}
c &= 2.998 \times 10^{10}\,\mathrm{cm/s} \\
k_B &= 8.62 \times 10^{-5}\,\mathrm{eV/K} \\
\hbar &= 6.582 \times 10^{-16}\,\mathrm{eV\,sec} \\
\hbar &= 1.973 \times 10^{-5}\,\mathrm{eV\,cm}/c \\
m_{\mathrm{Pl}} &= 1.221 \times 10^{19}\,\mathrm{GeV} = \sqrt{\frac{\hbar c}{G}} \\
T_0 &= 2.725\,\mathrm{K} = 2.348 \times 10^{-4}\,\mathrm{eV} \\
1\,\mathrm{Mpc} &= 3.086 \times 10^{24}\,\mathrm{cm} \\
H_0 &= 100h\,\mathrm{km/s/Mpc} \\
H_0 &= 3.24h \times 10^{-18}\,\mathrm{s}^{-1} \\
H_0/c &= 0.000334h\,\mathrm{Mpc}^{-1}
\end{aligned}
$$