UNIVERSITY of NORTH CAROLINA at CHAPEL HILL

Doctoral Written Examination in Physics, 2014

Part I: Classical Mechanics and Statistical Mechanics

Friday, May 9, 2014

Instructions: Please work in the assigned room, but take a break outside any time you wish. Mathematical handbooks and electronic calculators are allowed.

Begin each answer on a new sheet and write only on one side of each sheet.
Identify each sheet by:

Page Number_____ of Question_____ and your Identification Number______

CM: Classical Mechanics
Work 3 out of 5 problems

SM: Statistical Mechanics
Work 3 out of 5 problems

(Partial credit will be given for partial answers)

My work is completed in full observance of the Honor code:

Signature_________________________

Print name_______________________

UNIVERSITY of NORTH CAROLINA at CHAPEL HILL

Doctoral Written Examination in Physics, 2014

Part II: Electromagnetism I and Quantum Mechanics I

Monday, May 12, 2014

Instructions: Please work in the assigned room, but take a break outside any time you wish. Mathematical handbooks and electronic calculators are allowed.

Begin each answer on a new sheet and write only on one side of each sheet.
Identify each sheet by:

Page Number_____ of Question_____ and your Identification Number______

EMI: Electromagnetism I
Work 3 out of 5 problems

QMI: Quantum Mechanics I
Work 3 out of 5 problems

(Partial credit will be given for partial answers)

My work is completed in full observance of the Honor code:

Signature_________________________

Print name____________________________

UNIVERSITY of NORTH CAROLINA at CHAPEL HILL

Doctoral Written Examination in Physics, 2014

Part III: Electromagnetism II and Quantum Mechanics II

Monday, May 12, 2014

Instructions: Please work in the assigned room, but take a break outside any time you wish. Mathematical handbooks and electronic calculators are allowed.

Begin each answer on a new sheet and write only on one side of each sheet.
Identify each sheet by:

Page Number_____ of Question_____ and your Identification Number______

EMII: Electromagnetism II
Work 3 out of 5 problems

QMII: Quantum mechanics II
Work 3 out of 5 problems

(Partial credit will be given for partial answers)

My work is completed in full observance of the Honor code:

Signature_________________________

Print name____________________________

UNIVERSITY of NORTH CAROLINA at CHAPEL HILL

Doctoral Written Examination in Physics, 2014

Part III: Astro I (Stellar) and II (Cosmology)

Monday, May 12, 2014

Instructions: Please work in the assigned room, but take a break outside any time you wish. Mathematical handbooks and electronic calculators are allowed.

Begin each answer on a new sheet and write only on one side of each sheet.
Identify each sheet by:

Page Number_____ of Question_____ and your Identification Number______

Astro I (Stellar):
Work 3 out of 5 problems

Astro II (Cosmology):
Work 3 out of 5 problems

(Partial credit will be given for partial answers)

My work is completed in full observance of the Honor code:

Signature_________________________

Print name____________________________

CM-1

A mass m is positioned on a horizontal frictionless $x-y$ plane and attached to the vertices of an equilateral triangle (lying in the same plane) by three massless springs. Two of the springs have elastic constant k_1 while the third has k_2. Assume that the springs can be stretched or compressed but cannot be bent. None of the springs is deformed when the mass is in equilibrium at the geometrical center of the triangle.

(a) Expand the Hamiltonian to second order in terms of the small deviations about the equilibrium position and find the frequencies of the normal modes.

(b) Find the eigenvectors corresponding to these normal modes.

(c) What happens to the eigenvectors in the limit of $k_1 = k_2$?

Hint: Choose the coordinate system on the basis of the symmetry properties of the Hamiltonian.

CM-2

A particle of mass m slides without friction on the inner surface of a cone that has an opening angle of θ. The cone is standing vertically on its apex.

(a) Assuming uniform gravity and using standard spherical coordinates r, θ, ϕ, write down the Hamiltonian of the system.

(b) Assuming that the angle θ undergoes a slow change, find (approximately) the dependence of the particle's energy E upon θ under the condition $\tan\theta \gg gm^{1/2}l/E^{3/2}$ (i.e., very eccentric orbits on the cone), where l is the particle's angular momentum with respect to the vertical axis.

Hint: Compute the corresponding adiabatic invariant.

CM-3

The Hamiltonian of a (pseudo) relativistic one-dimensional oscillator is given by

$$H = \sqrt{c^2p^2 + m^2c^4} + \frac{m\omega^2x^2}{2}. \tag{1}$$

Expand this expression in powers of p/mc and compute the lowest order relativistic correction $\delta\omega$ to the frequency ω of the ordinary (non-relativistic) harmonic oscillator as a function of the system's energy E.
Hint: Introduce action-angle variables, solve the $0th$ order (non-relativistic) problem, and then use the $1st$ order classical perturbation theory.

CM-4

Consider a classical particle of mass m that is trapped in a one-dimensional quartic potential well

$$U(x) = \frac{1}{4}k\,x^4,$$

where k is a real-valued constant with appropriate units. From the symmetry of the well it is obvious that a particle that starts from rest at some point $x = a$ will undergo periodic motion around the origin.

(a) Find the period of this oscillation exactly.

(b) Interpret your results in the limit where a gets very small or very large.

You may use the following definite integral without proof

$$\int_0^a \frac{dx}{\sqrt{1-(x/a)^4}} = 1.31103\,a.$$

CM-5

Consider a free particle confined to a finite interval along the x axis (a one-dimensional box) moving rapidly and bouncing back and forth between the ends of the interval (i.e., box walls). Suppose that the particle begins with energy E_0 and the walls are very slowly brought toward each other, starting from a distance L_0 apart.

(a) Find the energy E of the particle in terms of the parameters given above when the box has width L.

(b) The one-dimensional analog of pressure is the time-averaged force $\bar{F}$ on a wall, where the average is over a reasonably long time (one period is sufficient for periodic motion). Find $\bar{F}$ when the box has width L. Hint: relate the average force exerted by the particle on a wall to the change in its momentum during a collision with the wall.

(c) In the adiabatic compression of a gas in three dimensions, PV^γ is a constant for some constant γ. In our one-dimensional one-particle "gas," $\bar{F}L^\gamma$ is essentially constant if the change in L is slow enough. Find γ for our case.

SM-1

Consider a system of N classical distinguishable harmonic oscillators where the Hamiltonian is given by

$$H = \sum_{i=1}^{N}\left(\frac{p_i^2}{2m} + \frac{1}{2}m\omega^2 q_i^2\right)$$

(a) Calculate $\Sigma(N,E)$, the total number of microstates with energy less or equal to E.

(b) Based on the calculated $\Sigma(N,E)$, show that $S(N,E) = Nk\left[1 + \ln\left(\frac{E}{N\hbar\omega}\right)\right]$.

Reminder: $\ln n! = n\ln n - n$; The volume of n-dim hypersphere of radius R is given by $V_n(R) = \frac{\pi^{n/2}}{(n/2)!}R^n = \frac{\pi^{n/2}}{(n/2)\Gamma(n/2)}R^n$ and $\Gamma(\nu) = (\nu-1)!$

SM-2

A particle can exist in any of the N equidistant states with energies ε, 2ε, 3ε, …, $N\varepsilon$.

(a) Compute its average energy $\langle E\rangle$ when it is in equilibrium with the thermal reservoir of temperature T.

(b) Find the value of $\langle E\rangle$ in the extreme $T \to 0$ and $T \to \infty$ limits.

SM-3

The eigenstate of a particle in a box is labeled by $|\varepsilon\rangle$ and its associated energy is ε. For N such non-interacting particles in the box of volume V and at temperature T, the partition function is given by

$$Q_N(V,T) = \sum_{\{n_{|\varepsilon\rangle}\}}{}' g\{n_{|\varepsilon\rangle}\} \exp\left(-\beta \sum_{|\varepsilon\rangle} n_{|\varepsilon\rangle}\varepsilon\right)$$

Here, $\beta = 1/(k_B T)$, $n_{|\varepsilon\rangle}$ is the number of particles in the single-particle state $|\varepsilon\rangle$, $\{n_{|\varepsilon\rangle}\}$ is a set of $n_{|\varepsilon\rangle}$ numbers and the summation is over all possible set of $\{n_{|\varepsilon\rangle}\}$ where $\sum_{|\varepsilon\rangle} n_{|\varepsilon\rangle} = N$, a condition indicated by the prime, and $g\{n_{|\varepsilon\rangle}\}$ is the number of quantum states of the N particles corresponding to the set $\{n_{|\varepsilon\rangle}\}$. For indistinguishable particles, the only possible values for $g\{n_{|\varepsilon\rangle}\}$ are either 0 or 1.

(a) Show that the grand partition function $\mathcal{Q}(\mu,V,T)$ is given by

$$\mathcal{Q}(\mu,V,T) = \prod_{|\varepsilon\rangle}\left(\sum_{n_{|\varepsilon\rangle}} \exp\left(\beta(\mu-\varepsilon)n_{|\varepsilon\rangle}\right)\right)$$

Here, the summation is over all $n_{|\varepsilon\rangle}$ values compatible with $g\{n_{|\varepsilon\rangle}\} = 1$. For instance, $n_{|\varepsilon\rangle} = 0 \text{ or } 1$ is compatible with $g\{n_{|\varepsilon\rangle}\} = 1$ for fermions.

(b) Now consider a hypothetical type of particles where $n_{|\varepsilon\rangle}$ values compatible with $g\{n_{|\varepsilon\rangle}\} = 1$ are 0, 1, 2. Show that the average value of $n_{|\varepsilon\rangle}$, namely the average occupation number of state $|\varepsilon\rangle$, is given by

$$< n_{|\varepsilon\rangle} >= \frac{1}{\exp\left[\beta(\varepsilon-\mu)\right]-1} - \frac{3}{\exp\left[3\beta(\varepsilon-\mu)\right]-1}$$

.

SM-4

If the atoms in a single square layer of side length L are on a square lattice (one atom per lattice site) and connected by effective spring constant κ, mechanical vibrations around their equilibrium positions give rise to normal modes called phonons. The atoms can move only within the plane. Invoke the Debye approximation that the frequency of oscillation is proportional to wave number $\omega(k) = ck$ (c is the speed of sound in the material and is assumed to be the same for both the longitudinal and the transverse modes) up to some cutoff value of k_D (assume this is given) that enforces the correct number of modes for the system ($\omega_D = ck_D$ is the Debye frequency).

(a) Calculate the number of modes or "states" up to frequency ω and give the expression of the total number of modes in terms of L and ω_D .

(b) Calculate the density of modes $D(\omega)$.

(c) From $D(\omega)$ calculate the internal energy stored in all of the modes making the approximation that the temperature is very low compared to the Debye frequency ($k_B T << \hbar\omega_D$).

(d) Calculate the specific heat C_V of the two-dimensional phonons, and speculate on the temperature dependence in one-dimensional crystals.

SM-5

Consider the following simple model of ferromagnetism, introduced by Weiss who postulated that the driving magnetic field within ferromagnets is given by

$$\vec{B}_{driving} = \vec{H} + \lambda \vec{M},$$

where $\vec{H}$ is the magnetic field and $\vec{M}$ is the magnetization, and $\lambda \gg 1$. We want to calculate the thermal average of the intrinsic magnetic moment $\langle \vec{\mu} \rangle$. Let us use the quantum mechanical distribution by supposing that the atomic magnetic moment $\vec{\mu}$ is either lined up parallel or anti-parallel to $\vec{B}_{driving}$ (which defines the z axis).

(a) Show that

$$\langle \mu_z \rangle = \mu(\tanh x),$$

where tanh is hyperbolic tangent function and $x = \frac{\mu}{kT}(H + \lambda M)$, with k being the Boltzmann constant.

(b) Show that there is a critical temperature (called the Curie temperature) given by $T_c = \frac{n\mu^2\lambda}{k}$ (where n is the density of atomic magnetic moments) such that for temperature $T < T_c$, a permanent magnetization can exist.

(c) Show that for sufficiently high temperature ($T \gg T_c$), the magnetic susceptibility is given by

$$\frac{M}{H} = \frac{1}{\lambda}\frac{T_c}{T - T_c}.$$

EMI-1

(a) Find the total charge, the dipole moment and the quadrupole moment of the charge density $\rho(\vec{r}) = -\vec{d} \cdot \vec{\nabla}\delta(\vec{r})$ where $\vec{d}$ is a constant vector. (For the dipole moment use the expression $\int (d\vec{r})\vec{r}\rho(\vec{r})$.)

(b) Repeat the calculation for the dipole moment by using the expression for multipole moments given by

$$\rho_{lm} = \int (d\vec{r}) r^l \sqrt{\frac{4\pi}{2l+1}} Y^*_{lm}(\theta,\phi)\rho(\vec{r}),$$

where Y_{lm} are the spherical harmonics.

(Recall that ρ_{1m} yields the dipole moment $\vec{p}$ as

$$\begin{aligned} \rho_{11} &= -\frac{1}{\sqrt{2}}(p_x - ip_y), \\ \rho_{10} &= p_z, \\ \rho_{1-1} &= \frac{1}{\sqrt{2}}(p_x + ip_y). \end{aligned}$$

Note that essentially all you need to know are the qualitative features of Y_{1m}: e.g., rY_{10} is proportional to z.)

EMI-2

(a) Consider a localized charge distribution with total charge e and electric dipole moment $\vec{d}$, etc, at the origin. The interaction energy of this charge distribution with an additional point charge e_1 located at a point $\vec{r}$ lying far outside the charge distribution is given by

$$E = \frac{ee_1}{r} + \vec{d} \cdot \frac{e_1\vec{r}}{r^3} + \ldots .$$

Use this expression to find the interaction energy for a dipole-dipole interaction with dipole $\vec{d_1}$ at the origin and dipole $\vec{d_2}$ at $\vec{r}$ (at zero temperature).

(b) Compute the thermal averaged interaction energy between the two dipoles in the high temperature limit.

EMI-3

Consider applying the Green's function technique to the electrostatic problem for a point charge (at $\vec{r'}$ with $z' > 0$) outside a uniform dielectric (with dielectric constant $\epsilon =$ constant) occupying the semi-infinite region $z < 0$, while the region $z > 0$ is a vacuum.
Assume that you have successfully calculated the Green's function to be given, for $z > 0$, by

$$G(\vec{r}, \vec{r'}) = \frac{1}{|\vec{r} - \vec{r'}|} - \frac{\epsilon - 1}{\epsilon + 1} \frac{1}{|\vec{r} - \vec{r''}|},$$

where $\vec{r''} = (x', y', z'')$ is the image point of $\vec{r'}$ with $z'' = -z'$. Finally recall that the total energy of a charge distribution in the presence of the dielectric medium

$$E = \frac{1}{2} \int (d\vec{r})(d\vec{r'}) \rho(\vec{r}) G(\vec{r}, \vec{r'}) \rho(\vec{r'}),$$

includes the mutual interactions of the charges.

(a) Find the force of attraction between a point charge (of charge e at $\vec{r}_0$ with $x_0 = 0, y_0 = 0, z_0 > 0$) and the dielectric medium.

(b) Replace the dielectric medium by a grounded conductor. Find the surface charge density induced on the conductor.

EMI-4

Consider three straight, infinitely long, parallel thin wires lying in a plane. Let the spacing between adjacent wires be d and assume that each wire carries a current I flowing in the same direction.

(a) Calculate the location of the two zeros in the magnetic field.

(b) If the middle wire is rigidly displaced a very small distance x ($x \ll d$) while the other two wires are held fixed, describe qualitatively the subsequent motion of the middle wire. Note: consider displacements x in the plane of the wires as well as off of the plane.

EMI-5

Consider a point charge in the exterior of a conducting or dielectric sphere.

(a) Easy part (20%): Use the method of images to find the electrostatic potential induced by a point charge q held at a distance $r > R$ from the center of a uniform conducting sphere of radius R. Assume the latter has a total charge Q and is insulated from any source of charge.

(b) Less-easy part (40%): The same method can be applied if the sphere is now dielectric with dielectric constant ϵ, but it is much harder to implement. Show an alternative way to solve the problem.

(c) Crucial question (40%): Connect the two methods above by explaining how you would obtain the image charge distribution inside the sphere from your solution of part (b).

QMI-1

(a) Write the quantum Hamiltonian for a particle of charge e and mass m subjected to the vector and scalar potentials of a static electromagnetic field. (Use $\hbar = c = 1$ throughout this problem.)

(b) Write the gauge transformation of a static electromagnetic potential, and show how the Schroedinger wavefunction of the Hamiltonian in part (a) transforms under this gauge transformation.

(c) Use the monopole potential with magnetic charge e_m,

$$A^I = \frac{e_m(1-\cos\theta)}{r\sin\theta}\hat{\phi} \quad \text{for the northern hemisphere } \theta < \pi - \epsilon,$$
$$A^{II} = -\frac{e_m(1+\cos\theta)}{r\sin\theta}\hat{\phi} \quad \text{for the southern hemisphere } \theta > \epsilon,$$

which has $\vec{B} = \frac{e_m}{r^2}\hat{r}$ everywhere on the sphere, to compute a quantization condition on the magnetic charge.

QMI-2

Consider two identical linear oscillators with spring constant k. The interaction potential is given by $H' = cx_1x_2$, where x_1 and x_2 are the oscillator variables. (Use $\hbar = c = 1$ throughout this problem.)

(a) Write the full Hamiltonian of this system with commutation relations $[x_i, p_j] = i\delta_{ij}$.

(b) Find the exact energy eigenvalues.

QMI-3

(a) Use the rule for addition of angular momentum to decompose the tensor product of two spin $\frac{1}{2}$ representations of the rotation group into a direct sum of irreducible representations, and give their dimensions. (Use $\hbar = c = 1$ throughout this problem.)

(b) Use Clebsch Gordan coefficients $\langle m_1, m_2 | j, m \rangle$ to express the eigenstates of the irreducible representations in the direct sum in part (a) as linear combinations of the eigenstates of the tensor product.

QMI-4

A system is in a state $|\Psi\rangle$ with an energy uncertainty

$$\Delta E \equiv \sqrt{\langle\Psi|\,(H - \overline{H})^2\,|\Psi\rangle}\,,$$

where $\overline{H} \equiv \langle\Psi|\,H\,|\Psi\rangle$.

(a) The state evolves for a short time Δt. Find the probability, to second order in Δt, that the system is still in the state $|\Psi\rangle$ at time Δt.

(b) Let the system be a spin-1/2 particle with

$$H = a\sigma_z,$$

and

$$|\Psi\rangle = |S_x+\rangle\ .$$

What is the probability of that the system will still be found in state $|\Psi\rangle$ at an *arbitrary* later time t?

(c) Let $t = \Delta t$ and expand the result of part (b) to second order in Δt. Show that your result from part (a) is correct for this specific example.

QMI-5

Consider two interacting particles. The first particle (1) has spin-1 and the second particle (2) has spin-1/2.

(a) What are the possible values for the total angular momentum of the two particle system? How many independent (spin) states does the system have?

(b) Suppose the two particles interact via the Hamiltonian

$$H = \frac{a}{\hbar^2}\vec{S}_1 \cdot \vec{S}_2 + \frac{b}{\hbar}(S_{1z} + S_{2z})\,.$$

Find all the eigenvalues of the Hamiltonian.

EMII-1

In the electric dipole approximation the angular distribution of radiated power is

$$\frac{dP}{d\Omega} = \frac{c}{8\pi}\mathrm{Re}\left[r^2\vec{n}\cdot(\vec{E}\times\vec{B}^*)\right] = \frac{c}{8\pi}k^4|(\vec{n}\times\vec{d})\times\vec{n}|^2,$$

for a harmonically oscillating dipole with a complex vector amplitude $\vec{d}$ and with $k = \omega/c$.

Assume that a plane wave of light with frequency ω is propagating in the z direction (i.e., along $\vec{e}_3$ unit vector). Let the light scatter from an electron with mass m and charge $-e$.

(a) Assume the light is unpolarized. Pick a representation for the electric field and discuss how the lack of polarization affects the calculation of scattered power.

(b) Let θ be the polar angle relative to the z axis and let ϕ be an azimuthal angle around this axis. Calculate the angular dependence of scattered power, $dP/d\Omega$.

(c) Without necessarily doing a detailed calculation, explain the nature of the polarization of the scattered light and how it varies as a function of θ.

EMII-2

A relativistic electron with Lorentz factor γ strikes a stationary electron. After the event the two electrons are observed to emerge from the collision along with a newly created muon-antimuon pair,

$$e^- + e^- \to e^- + e^- + \mu^+ + \mu^- .$$

(a) Assume the pair creation reaction is just at threshold. Calculate an expression for the required value of the Lorentz factor γ in terms of the muon m_μ and electron m_e masses.

(b) Find an expression for the Lorentz factor γ' of the four emerging particles in terms of m_μ and m_e.

(c) Given that $m_\mu = 105.7$ MeV and $m_e = 0.5110$ MeV, compute γ and γ'.

EMII-3

A stationary magnetic dipole, $\vec{m} = m\vec{k}$, is situated above an infinite uniform surface current, $\vec{K} = K\vec{i}$ in the laboratory frame S, where $\vec{i}$ and $\vec{k}$ are unit vectors pointing along the +x and +z directions, respectively.

Suppose that the surface current consists of a uniform surface charge σ, moving at velocity $\vec{v} = v\vec{i}$. Suppose further that the magnetic dipole consists of a uniform line charge λ, circulating at speed v (same v) around a square loop of side length l, with two sides parallel to the x-axis and the other two parallel to the y-axis.

Examine this configuration from the point of view of the frame S' moving in the +x direction at speed v (relative to the laboratory frame). Show that in the S' frame the current loop carries an electric dipole moment, and calculate the resulting torque on the electric dipole.

EMII-4

The spectral-angular distribution of energy radiated by a single relativistic electron can be computed from

$$\frac{dW}{d\omega d\Omega} = \frac{e^2\omega^2}{4\pi^2 c}\left|\int dt'\vec{n} \times (\vec{n} \times \vec{\beta}) \exp[i\omega(t' - \vec{n} \cdot \vec{x}(t')/c)]\right|^2 .$$

Consider a nucleus that undergoes double beta decay. Assume two relativistic electrons simultaneously emerge from the nucleus and fly away in opposite directions with the same energy.

Let one electron have velocity $\vec{\beta}_1 = (0, 0, \beta)$ and position vector $\vec{x}_1(t') = (0, 0, c\beta t')$ for $t' > 0$. The other electron's velocity $\vec{\beta}_2$ and position vector $\vec{x}_2$ are similar but with the sign of β reversed. Let the observer be at $\vec{n} = (\sin\theta, 0, \cos\theta)$.

(a) Calculate $dW/d\omega d\Omega$ to find the radiative correction to double beta decay.

(b) Compare the result to that if only one electron appeared (regular beta decay).

(c) Consider the nonrelativistic limit of part (a) (i.e., $\kappa \to 1$). What does the β and angular dependence imply in terms of multipolar radiation?

EMII-5

A particle moves at relativistic velocity $v = \beta c$ along the x axis in the lab frame. In its own rest frame (primed frame) it emits photons with an angular distribution

$$\frac{dN}{d\Omega'} = f(\theta', \phi'),$$

where θ' is the angle of emission relative to the x' axis, and ϕ' is the azimuthal angle about the x' axis.

(a) Use the Lorentz transformation of the wave vector k'^{α} of an emitted photon to show that, for a photon emitted at the angle θ' in the primed frame, the angle of emission in the laboratory frame satisfies

$$\cos\theta' = \frac{\cos\theta - \beta}{1 - \beta\cos\theta}.$$

(b) Use this result to show that the distribution of photons in the laboratory frame is

$$\frac{dN}{d\Omega}(\theta, \phi) = \frac{f(\theta', \phi')}{\gamma^2(1 - \beta\cos\theta)^2}.$$

(c) Finally, assume that the angular distribution of photon number is isotropic in the primed frame. Furthermore, assume that the Lorentz factor connecting the primed frame and the lab frame is large, $\gamma \gg 1$. Evaluate in terms of γ the ratio between the forward ($\theta = 0$) and reverse ($\theta = \pi$) angular distributions of photons as seen in the lab frame.

QMII-1

Consider the excited level $n = 2$ of a helium ion (a hydrogen-like system with one electron and two protons). We apply a weak uniform electric field in the z direction, with the perturbation operator $V = -ez|\mathbf{E}|$.

(a) Use symmetries to argue that 14 of the 16 matrix elements of V vanish.

(b) Use degenerate perturbation theory to compute the zeroth-order eigenstates and first-order energy shifts. You might need:

$$
\begin{aligned}
Y_0^0 &= \frac{1}{\sqrt{4\pi}}\,, \quad Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta\,, \quad Y_1^{\pm 1} = \pm\sqrt{\frac{3}{8\pi}}(\sin\theta)e^{\pm i\phi} \\
R_{20}(r) &= \left(\frac{Z}{2a_0}\right)^{3/2}\left(2 - \frac{Zr}{a_0}\right)e^{-Zr/(2a_0)}\,, \quad R_{21}(r) = \left(\frac{Z}{2a_0}\right)^{3/2}\frac{Zr}{\sqrt{3}a_0}e^{-Zr/(2a_0)}
\end{aligned}
$$

QMII-2

An electron is subjected to a spin-orbit interaction

$$H = \frac{1}{2m_e^2c^2}\frac{1}{r}\frac{dV_c}{dr}(\mathbf{L}\cdot\mathbf{S})$$

with a shielded Coulomb potential $V_c = -V_0 e^{-\mu r}/(\mu r)$. Assume this is sufficiently close to the Coulomb problem for small radii that you can use hydrogen wavefunctions for the unperturbed problem.

(a) Compute the energy shift of the $n = 2, \ell = 1$ state for spin up ($j = \ell + 1/2$), leaving the radial integral I_r unperformed.

(b) Now perform the radial integral. You may use R_{21} from above. Give the energy shift in the limit $\mu a_0 \ll 1$, using an expression for the ratio V_0/μ from comparing to the Coulomb problem.

QMII-3

Consider a nucleus in a magnetic field with both a constant field in the z direction and an oscillating field in the xy plane, giving rise to an unperturbed Hamiltonian H_0 and a perturbation $V(t)$:

$$H_0 = E_0 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \quad V(t) = E_1 \begin{pmatrix} 0 & e^{i\omega t} \\ e^{-i\omega t} & \end{pmatrix} \tag{1}$$

with E_0 and E_1 constant. Assume that the frequency is tuned to near resonance: $\omega \approx \omega_{12}$.

(a) If the nuclear spin is up at time $t = 0$, compute the probability that the nucleus has spin down at time t as a function of time.

(b) At what time t_1 are the probabilities of spin up and spin down equal? Semiclassically, where is this spin pointing?

You might use

$$i\hbar \frac{d}{dt} c_n(t) = \sum_m V_{nm} e^{i\omega_{mn}t} c_m(t) \ .$$

QMII-4

Consider a one dimensional chain of N regularly spaced atoms. Suppose each positively charged atomic core acts on the electron as a delta function well. This situation can be described by a periodic potential $V(x)$, such that $V(x+a) = V(x)$, of delta function wells:

$$V(x) = -\alpha \sum_{j=0}^{N-1} \delta(x - ja)$$

(a "Dirac comb"). In between the delta functions, $V(x) = 0$ for $0 < x < a$.

Imposing periodic boundary conditions for the wavefunction, $\psi(x=0) = \psi(x=Na)$, estimate the allowed energies and find solutions for the wavefunction. Describe the symmetry considerations used in your derivation.

QMII-5

Use the first Born approximation to answer the following questions:

(a) Find the differential and total scattering cross sections for an incident particle of mass m and momentum $\mathbf{k}$ for a Dirac potential given as:

$$V(\mathbf{r}) = g\delta^3(\mathbf{r}),$$

where g is a positive real constant with appropriate units.

(b) Now consider the following variation:

$$V(\mathbf{r}) = g(\delta^3(\mathbf{r} + a\hat{z}) + \delta^3(\mathbf{r} - a\hat{z})),$$

where a is a positive real constant with appropriate units. Using physical arguments only, find the total scattering cross sections for the two separate the cases where $ak \ll 1$ and $ak \gg 1$. Provide a physical interpretation of these two cases.

(c) Calculate the differential scattering cross section for the potential in (b) and show that the total cross-section has the asymptotic behavior that you found in the previous part. You might find the following integral useful: $\int dx \, \cos^2(ax) = \frac{1}{4a}\sin(2ax) + \frac{x}{2}$.

ASI-1

The following questions concern fusion in double-shell-burning sources.

(a) Using the attached table of atomic masses, compute the energy released per baryon during the fusion of hydrogen into helium. Assume that the total loss in neutrinos for the process is 0.51 MeV. Do the same for the fusion of helium into carbon.

(b) Write an expression for the rate at which a nuclear-burning shell advances into the material above it.

(c) Define the condition under which the helium-burning shell will advance closer to the hydrogen-burning shell.

ASI-2

Consider opacity and the curve of growth.

(a) For stars with temperatures and chemical compositions similar to the Sun, what is the primary source of continuous opacity in the visible portion of the spectrum?

(b) Using the Saha equation, it can be shown that in the Sun, the dominant ionization of iron is its singly ionized state, Fe II. Fe I abundance are much lower. Show that the Fe I lines are insensitive to pressure.

(c) Show the pressure dependence of the Fe II lines.

(d) Sketch a curve of growth for an Fe II line, qualitatively. Specifically plot the abundance as log A vs. the log of the reduced equivalent width, $\log(W_\lambda / \lambda)$ for $\log g = 2$ and $\log g = 4.5$.

ASI-3

Consider convection in a stellar interior.

(a) Consider a bubble of gas within a much larger layer. Derive an expression assuming adiabatic conditions that may be used to assess whether convection will occur; that is, that the bubble becomes more buoyant as it rises.

(b) Recast that result assuming the perfect gas law prevails into a relation involving the ratio of the specific heats, ϒ, and the two derivatives, $d \log T/d \log P$ and $d \log \mu/ d \log P$, where μ is the mean molecular weight.

(c) Under what conditions might $d \log \mu / d \log P$ become important?

ASI-4

The following questions address neutron-capture nucleosynthesis.

(a) Consider the attached figure, showing all stable isotopes of Rb, Sr, Zr, Nb, Mo, Tc, and Ru in boxes. Those with open circles have half-lives to β-decay of at least 10^4 years. Empty boxes are unstable to β-decay and have very short half-lives. Identify the isotopes of Zr, Nb, Mo, Tc, and Ru that can be produced only by the r-process.

(b) The solar mass fraction of ^{96}Mo is $X_\odot = 1.111 \times 10^{-9}$, and it has an average neutron-capture cross section of $\langle\sigma\rangle = 112 \pm 8$ mb at T = 30 keV. Similarly, for ^{100}Ru, $X_\odot = 6.097 \times 10^{-10}$, and $\langle\sigma\rangle = 206 \pm 13$ mb at T = 30 keV. Verify that the "local approximation" for the s-process holds in this mass region.

(c) Use this to estimate the r-process contribution to ^{98}Mo, given that $X_\odot = 1.659 \times 10^{-9}$ and $\langle\sigma\rangle = 99 \pm 7$ mb, also at T = 30 keV.

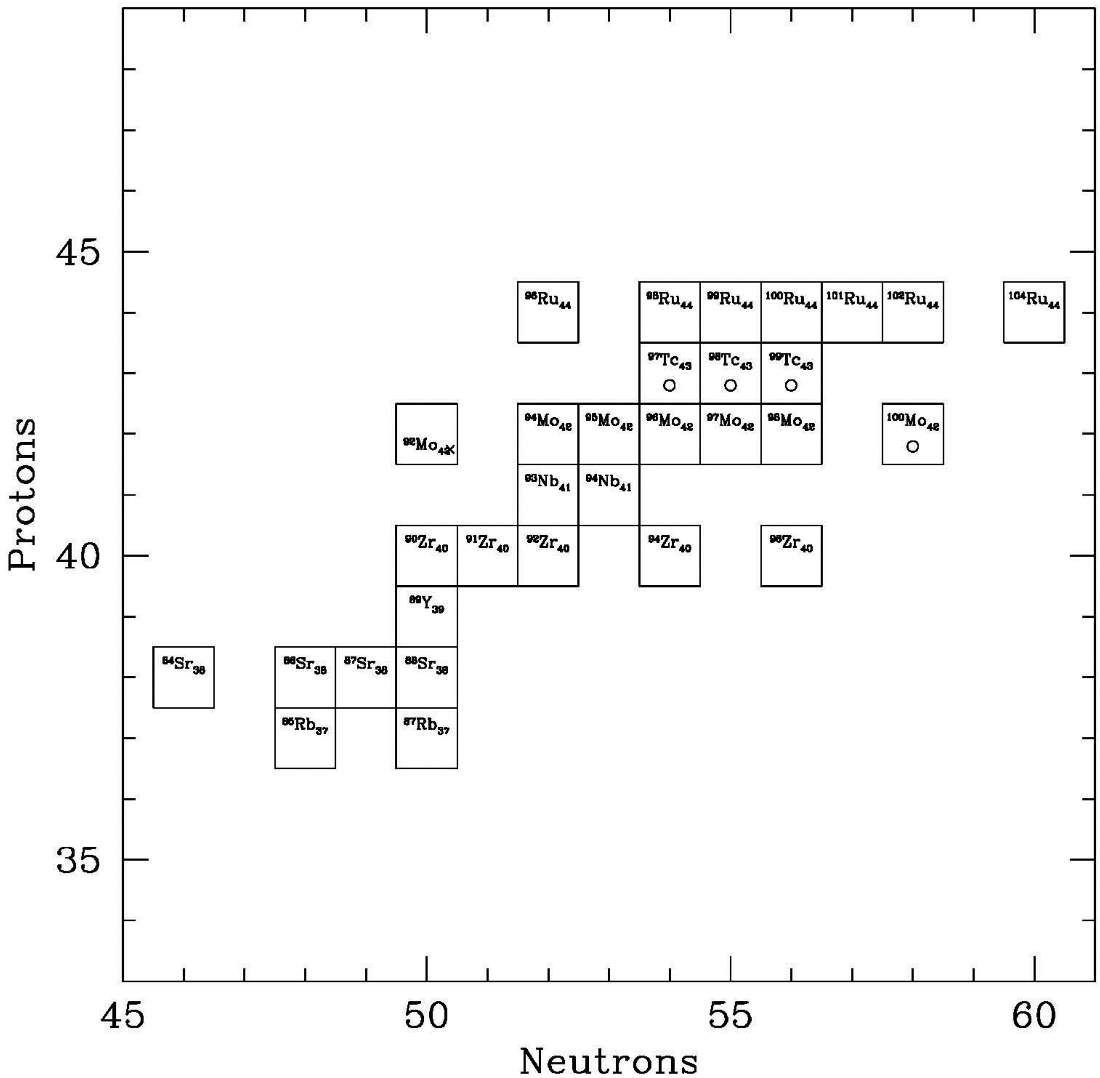
ASI-5

This question relates to continuum opacities in stars.

Show how a discontinuity in the continuum spectrum due to atomic hydrogen over wavelength range 0.3 – 1 µm can be used to measure n_e of F or G main sequence stars if T_{eff} can be obtained by other means.

Species	Atomic mass (in units of u[#])
e^-	5.4857990×10^{-4}
n	1.0086649
p	1.0072765
^{1}H	1.0078250
^{2}H	2.0141018
^{3}H	3.0160493
^{3}He	3.0160293
^{4}He	4.0026032
^{6}Li	6.0151223
^{7}Li	7.0160040
^{8}Be	8.0053051
^{9}Be	9.0121821
^{12}C	12.000000
^{13}C	13.003355
^{14}N	14.003074
^{15}N	15.000109
^{16}O	15.994915
^{17}O	16.999132
^{18}O	17.999161
^{20}Ne	19.992440
^{24}Mg	23.985042
^{28}Si	27.976927
^{32}S	31.972071
^{56}Fe	55.934942

[#] u = atomic mass unit = 1.661×10^{-24} g = 931.5 MeV/c^2.



Protons
Neutrons
35
40
45
45
50
55
60
$^{96}Ru_{44}$
$^{98}Ru_{44}$
$^{99}Ru_{44}$
$^{100}Ru_{44}$
$^{101}Ru_{44}$
$^{102}Ru_{44}$
$^{104}Ru_{44}$
$^{97}Tc_{43}$
$^{98}Tc_{43}$
$^{99}Tc_{43}$
$^{92}Mo_{42}$
$^{94}Mo_{42}$
$^{95}Mo_{42}$
$^{96}Mo_{42}$
$^{97}Mo_{42}$
$^{98}Mo_{42}$
$^{100}Mo_{42}$
$^{93}Nb_{41}$
$^{94}Nb_{41}$
$^{90}Zr_{40}$
$^{91}Zr_{40}$
$^{92}Zr_{40}$
$^{94}Zr_{40}$
$^{96}Zr_{40}$
$^{89}Y_{39}$
$^{84}Sr_{38}$
$^{86}Sr_{38}$
$^{87}Sr_{38}$
$^{88}Sr_{38}$
$^{85}Rb_{37}$
$^{87}Rb_{37}$

Speed of light	c	$2.99792458 \times 10^{10}$ cm/s
Gravitational constant	G	6.67259×10^{-8} cm^3/g/ s^2
Planck constant	h	$6.6260755 \times 10^{-27}$ erg s
Boltzmann constant	k	1.380658×10^{-16} erg/K
		8.617385×10^{-5} eV/K
Elementary charge	e	$4.8032068 \times 10^{-10}$ esu
Atomic mass unit	u	1.660540×10^{-24} g
		931.5 MeV/c^2
Mass of electron	m_e	$9.1093897 \times 10^{-28}$ g
		0.511 MeV/c^2
Mass of proton	m_p	$1.6726231 \times 10^{-24}$ g
		938.3 MeV/c^2
Mass of neutron	m_n	$1.6749286 \times 10^{-24}$ g
		939.6 MeV/c^2
Mass of ^{1}H atom	m_H	$1.6735344 \times 10^{-24}$ g
Stefan–Boltzmann constant	σ	5.67051×10^{-5} erg/cm^2/K^4/s
Thomson scattering constant	σ_T	6.6524×10^{-25} cm^2

Solar mass	$M_\odot$	1.9891×10^{33} g
Solar radius	$R_\odot$	6.95508×10^{10} cm
Solar luminosity	$L_\odot$	3.8458×10^{33} erg/s
Solar effective temperature	$T_{eff\odot}$	5777 K
Earth's mass	$M_\oplus$	5.9742×10^{27} g
Earth's radius	$R_\oplus$	6.378136×10^{8} cm
Astronomical unit	AU	$1.4959787066 \times 10^{13}$ cm
Light-year	ly	$9.460730472 \times 10^{17}$ cm
Parsec	pc	3.0856776×10^{18} cm
		3.26167 ly

ASII-1

In this problem assume that there is some unknown number N_ν of neutrino (lepton) flavors. You can assume that all the neutrinos are massless and that they decoupled from all other particles when the radiation temperature cooled to 1 MeV.

(a) At temperatures below 0.5 MeV (after electron-positron annihilation was complete), the neutrino and photon temperatures settled into a ratio of

$$\frac{T_\nu}{T_\gamma} = \left(\frac{4}{11}\right)^{1/3}.$$

Explain and show by calculation the origin of this number.

(b) The energy density of the photons alone is given by $\rho_\gamma = a_{bb}T^4$, where $a_{bb} = \pi^2 k^4/15c^3\hbar^3$ is the black-body constant. With a radiation fluid that is composed of only photons and neutrinos after electron-positron annihilation, the total radiation energy density can be written as a factor $1+F$ times that of the photons alone,

$$\rho_{\rm rad} = \rho_\gamma + \rho_\nu = a_{bb}T_\gamma^4(1+F).$$

Incorporating the number of neutrino flavors N_ν, determine the expression for F.

ASII-2

(a) What would the primordial helium fraction ($Y \equiv 4n_{\rm He}/n_b$) be if the mass difference between the neutron and proton was 0.647 MeV, i.e. half its actual value? The neutron lifetime is 886 seconds. Assume that protons and neutrons froze-out at a temperature of $T = 0.8$ MeV and that helium formed 270 seconds later when the temperature was 0.07 MeV. Compare your result to the observed primordial helium fraction ($Y_{\rm obs} = 0.25 \pm 0.01$).

(b) Consider two dark matter candidates X and Y. Particle X has twice the mass of particle Y, and its velocity-averaged annihilation cross section is half of Y's velocity-averaged annihilation cross section. If both particles are thermal relics, what is the present-day ratio ρ_X/ρ_Y? Assume that both particles have a freeze-out temperature that is 0.1 times their mass.

ASII-3

In a radiation dominated universe, the temperature-time relation is

$$\frac{T}{1\times 10^{16}\mathrm{K}} = \left(\frac{3.155\times 10^{-13}\,\mathrm{sec}}{t}\right)^{1/2}$$

for $T > 10^{16}$ K, assuming that $g_* = g_{*S} = 106.75$ at these high temperatures.

(a) Suppose that the universe is open (negatively curved) and rapidly becomes dominated by its curvature soon after the temperature has fallen to 3×10^{25} K. How is the above relation altered? Calculate the age of the universe in this case when the temperature has fallen to 3 K.

(b) Suppose a flat universe that contains only radiation and a cosmological constant with $\Omega_\Lambda = 0.7$ today. Compute the deceleration parameter

$$q \equiv -\frac{\ddot{a}a}{\dot{a}^2},$$

where a dot denotes differentiation with respect to proper time t, as a function of radiation energy density $\Omega_r(z) = \rho_r(z)/\rho_{\mathrm{crit}}(z)$ and calculate the redshift at which q changes sign.

ASII-4

(a) Derive the following relationship between the observed CMB temperature fluctuations on scales greater than 1° and the matter density fluctuation at the time of recombination (τ_*) for adiabatic perturbations:

$$\frac{\Delta T}{T}(\tau_0) \simeq \Theta_0(\tau_*) + \Psi(\tau_*) \simeq -\frac{1}{6}\delta_m(\tau_*).$$

(b) Qualitatively describe three ways the CMB angular power spectrum changes if the density of baryons ($\Omega_b h^2$) increases.

ASII-5

In the future, dark energy will dominate the Universe. Assume that dark energy has a **constant** equation of state parameter w ($w = P/\rho$) with $-0.9 > w > -1$.

(a) What is $\epsilon_H \equiv -\dot{H}/H^2$ in terms of w when dark energy dominates the Universe? (A dot denotes differentiation with respect to proper time t.)

(b) Evaluate the dimensionless power spectrum of the curvature fluctuations that are generated during the dark-energy dominated era for modes that exit the horizon when $a = 10^5$ (for $a = 1$ today). Compare these perturbations to the perturbations that were generated during inflation.

(c) Evaluate the scalar spectral index

$$n_s = 1 + \frac{d \ln \mathcal{P}_\zeta}{d \ln k}$$

for the perturbations generated during the dark-energy dominated era. Provide a physical explanation for this red or blue tilt.

Potentially Useful Formulae and Constants

$$c = 2.998 \times 10^{8}\,\mathrm{m/s}$$
$$k_B = 8.62 \times 10^{-5}\,\mathrm{eV/K}$$
$$\hbar = 6.58 \times 10^{-16}\,\mathrm{eV\,sec}$$
$$m_{\mathrm{Pl}} = 1.221 \times 10^{19}\,\mathrm{GeV} = \sqrt{\frac{\hbar c}{G}}$$
$$m_e = 0.511\,\mathrm{MeV}$$
$$m_p = 938\,\mathrm{MeV}$$
$$m_n = m_p + 1.2933\,\mathrm{MeV}$$
$$H_0 = 2.133h \times 10^{-42}\,\mathrm{GeV}$$
$$f = \frac{1}{e^{(E-\mu)/T} \pm 1}$$
$$f_\gamma = \frac{1}{e^{p/[T(1+\Theta)]} - 1}$$
$$n_{\mathrm{eq}} = g\left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T} \quad \text{for } T \ll m$$
$$\mathcal{P}_\zeta(k) = \frac{1}{\pi\epsilon_H}\left(\frac{H}{m_{\mathrm{Pl}}}\right)^2\Bigg|_{k=aH}$$
$$\mathcal{P}_h(k) = \frac{4}{\pi}\left(\frac{H}{m_{\mathrm{Pl}}}\right)^2\Bigg|_{k=aH}$$
$$r \equiv \frac{4\mathcal{P}_h(k)}{\mathcal{P}_\zeta(k)}$$

$$ds^2 = -(1+2\Psi)dt^2 + a^2(t)(1+2\Phi)\delta_{ij}dx^i dx^j = a^2(\tau)\left[-(1+2\Psi)d\tau^2 + \delta_{ij}(1+2\Phi)dx^i dx^j\right]$$

$$\frac{d}{d\tau}\delta_m + ikv_m + 3\frac{d}{d\tau}\Phi = 0$$
$$\frac{d}{d\tau}\Theta_0 + k\Theta_1 + \frac{d}{d\tau}\Phi = 0$$
$$k^2\Phi + 3\frac{1}{a}\frac{da}{d\tau}\left(\frac{d}{d\tau}\Phi - \Psi\frac{1}{a}\frac{da}{d\tau}\right) = 4\pi Ga^2\left[\delta_m\rho_m + 4\rho_r\Theta\right]$$
$$k^2(\Phi+\Psi) = -32\pi Ga^2\left[\rho_r\Theta_2 + \rho_\nu\mathcal{N}_2\right]$$