UNIVERSITY of NORTH CAROLINA at CHAPEL HILL

Doctoral Written Examination in Physics, 2013

Part I: Classical mechanics and Statistical mechanics

Saturday, May 11, 2013

Instructions: Please work in the assigned room, but take a break outside anytime you want to. Mathematical handbooks and electronic calculators are allowed.
Begin each answer on a new sheet and write <u>only</u> on <u>one side</u> of each sheet.
Identify each sheet by:

Page_____of Question:_____ The number (1-16) from the front page:____

CM: Classical Mechanics
Work out 3 out of 5 problems

SM: Statistical Mechanics
Work out 3 out of 5 problems

(Partial credit will be given for partial answers)

My work is completed in full observance of the Honor code:

Signature________________________

Print name______________________

UNIVERSITY of NORTH CAROLINA at CHAPEL HILL

Doctoral Written Examination in Physics, 2013

Part II: Electromagnetism I and Quantum mechanics I

Monday, May 13, 2013

Instructions: Please work in the assigned room, but take a break outside anytime you want to. Mathematical handbooks and electronic calculators are allowed.
Begin each answer on a new sheet and write only on one side of each sheet.
Identify each sheet by:

Page_____of Question:_____ The number (1-16) from the front page:____

EMI: Electromagnetism I
Work out 3 out of 5 problems

QMI: Quantum Mechanics I
Work out 3 out of 5 problems

(Partial credit will be given for partial answers)

My work is completed in full observance of the Honor code:

Signature_________________________

Print name____________________________

UNIVERSITY of NORTH CAROLINA at CHAPEL HILL

Doctoral Written Examination in Physics, 2013

Part III: Electromagnetism II and Quantum mechanics II

Monday, May 13, 2013

Instructions: Please work in the assigned room, but take a break outside anytime you want to. Mathematical handbooks and electronic calculators are allowed.
Begin each answer on a new sheet and write <u>only</u> on <u>one side</u> of each sheet.
Identify each sheet by:

Page_____of Question:_____ The number (1-16) from the front page:____

EMII: Electromagnetism II
Work out 3 out of 5 problems

QMII: Quantum mechanics II
Work out 3 out of 5 problems

(Partial credit will be given for partial answers)

My work is completed in full observance of the Honor code:

Signature_________________________

Print name____________________________

UNIVERSITY of NORTH CAROLINA at CHAPEL HILL

Doctoral Written Examination in Physics, 2013

Part III: Astro I and II

Monday, May 13, 2013

Instructions: Please work in the assigned room, but take a break outside anytime you want to. Mathematical handbooks and electronic calculators are allowed.
Begin each answer on a new sheet and write only on one side of each sheet.
Identify each sheet by:

Page_____of Question:_____ The number (1-16) from the front page:____

Astro I:
Work out 3 out of 5 problems

Astro II:
Work out 3 out of 5 problems

(Partial credit will be given for partial answers)

My work is completed in full observance of the Honor code:

Signature_________________________

Print name____________________________

CM-1

A massless string of fixed total length l passes through a hole in a frictionless table. A point mass m_1 is connected to the string and slides without friction on the table. A second point mass m_2 hangs vertically below the table from the other end of the string. The first mass moves in two dimensions but never leaves the surface of the table and the second mass is only allowed to move up or down.

(a) Write the Lagrangian for the system and find the equations of motion. Describe the symmetries of the Lagrangian, the conserved quantities, and their physical interpretations.

(b) Give the conditions for m_1 to remain in stationary orbit with radius a. Find the effective potential for radial motion and sketch it.

(c) Starting with this stationary state, if the hanging mass is pulled down slightly and released it will begin to oscillate vertically. What quantity is conserved in the process of perturbing the system this way?

(d) Solve for small amplitude radial motion of the perturbed stationary state. What is the frequency of this motion? Compare it to the frequency of circular motion in the unperturbed state.

CM-2

A uniform stick of length $2a$ and mass m leans against a frictionless vertical wall and stands on a frictionless horizontal surface. The initial angle between the stick and the floor is θ_0. When the stick is released it will slide down under the influence of the uniform gravitational field (g). For a period of time the stick will remain in contact with the vertical wall. Note: The moment of inertia of the stick about its center of mass is $I = (1/3)ma^2$.

(a) Obtain an expression for the time t that is required for the stick to reach any new angle $\theta < \theta_0$ (prior to loss of contact with the vertical wall). You may leave the expression in the form of an integral, $t = \int f(\theta, \theta_0)\, d\theta$.

(b) Given the initial angle θ_0, find the angle θ_1 at which the upper end of the stick leaves the vertical wall.

CM-3

1. Stability of rotation:

   (a) Write down Euler's equations in the absence of torque.

   (b) Use these equations to derive two equations of the form

   $$\ddot{\omega}_1 = f(\omega_2, \omega_3)\omega_1 \,, \quad \ddot{\omega}_2 = g(\omega_1, \omega_3)\omega_2 \,.$$

   What are f and g?

   (c) Suppose the object is rotating almost completely around the "3" axis, so that ω_3 is large (you can take it to be constant) and ω_1, ω_2 are small.

      i. Now suppose in addition that I_3 is the largest of the moments of inertia. What can you say (approximately) about the behavior of ω_1 and ω_2 as a function of time? What does this mean about the stability of the motion?

      ii. What if I_3 is the smallest moment?

      iii. And what if I_3 is between I_1 and I_2?

CM-4

Consider a transformation for a one-dimensional system

$$Q = q\cos\phi - p\sin\phi, \qquad P = q\sin\phi + p\cos\phi$$

(a) Are there any values of the parameter ϕ for which this transformation appears to be canonical?
(b) If so, find the corresponding type-I generating function $F_1(q, Q, t)$, such that $p = \partial F_1/\partial q$ and $P = -\partial F_1/\partial Q$.
(c) Provide an example of the Hamiltonian which has the same functional form in terms of Q and P as it does in terms of q and p.
(d) How can this invariance be used to find the action-angle variables for such a Hamiltonian?

CM-5

(CM-5) The motion of a particle of mass m is confined to the curve in the vertical $x - z$ plane parametrized by the angle ϕ as follows:

$$x = l(2\phi + \sin 2\phi), \qquad z = l(1 - \cos 2\phi)$$

(a) Derive the Hamiltonian of the system $H(\phi, p_\phi)$,
(b) Construct the explicit integral form of the solution $S(\phi, E, t)$, where E is energy, of the corresponding Hamilton-Jacobi equation.
(c) Obtain the integral equation for the system's trajectory $\phi(t)$ and describe the corresponding type of motion;
(d) Find the value of the action variable $J = \oint p_\phi d\phi$ in terms of E, l, and g (free fall acceleration);
(e) How would the energy E of the system change under an adiabatic evolution of the parameter l?

SM-1

A container of volume V held at temperature T contains N indistinguishable molecules (assume no internal degree of freedom), some (N_{ad}) are adsorbed at the adsorption sites on the internal surface of the container walls and others ($N_{gas} = N - N_{ad}$) are in the gas phase (considered as an ideal gas) confined by the container. The total number of adsorption sites is N_0. Each adsorption site can accommodate at most one molecule ($N_{ad} \le N_0$). Molecules adsorbed at different sites do not interact with each other but can exchange with each other and move to different adsorption sites. The chemical potential of an ideal gas is given by $\mu_{gas} = k_B T \ln\left[\frac{N_{gas}}{V}\left(\frac{h^2}{2\pi m k_B T}\right)^{3/2}\right]$ where h is the Planck constant, m is the mass of the molecule, and k_B is the Boltzmann constant. Show that N_{ad} is determined by

$$\frac{N - N_{ad}}{V}\left(\frac{h^2}{2\pi m k_B T}\right)^{3/2} = \frac{N_{ad}}{(N_0 - N_{ad})\,a(T)}$$

where $a(T)$ is the partition function of a single adsorbed molecule and $\ln N! = N \ln N - N$.

SM-2

As you might remember, for an ideal Bose gas

$$\frac{PV}{kT} = \ln Q(z,V,T) = -\sum_s \ln\left(1 - z e^{-\varepsilon_s/kT}\right) \quad \text{where } z = e^{\mu/kT}.$$

For a photon gas in a cavity of volume V the density of states is given by

$$a(\varepsilon) = \frac{V}{h^3}\frac{\partial}{\partial\varepsilon}\left(g\frac{4}{3}\pi p^3\right) \text{ and } g = 2 \text{ for the two polarizations and } \varepsilon = pc$$

For photon gas, $\mu = 0$ and $z=1$.

Show that $S = V\dfrac{32\pi^5 k^4}{45(hc)^3}T^3$.

Reminder: $\int_0^\infty \frac{x^3}{e^x - 1}dx = \frac{\pi^4}{15}$ and

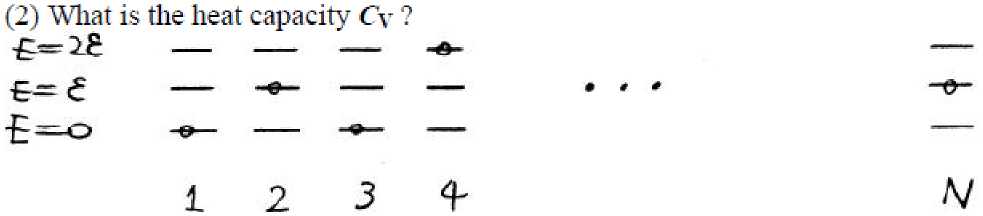
$\Phi = E - TS - \mu N = -PV$, $d\Phi = -SdT - PdV - Nd\mu = -SdT - PdV$ since $\mu = 0$.

SM-3

Consider N distinguishable molecules, each can be in one of the three energy states, E = 0; ε; 2ε.

(1) What is the Helmholtz free energy, energy and the entropy of the system at temperature T ?

(2) What is the heat capacity C_V ?



SM-4

Consider the Ising model $H=-J\sum_{n.n.}\sigma_i\sigma_j - B\mu\sum_i \sigma_i$ where each of the N spins with magnetic moment μ has q nearest neighbors ($n.n.$). N_{++} , N_{--} , N_{+-} denote the number of up-up, down-down, and opposite signs nearest-neighbor spin pairs, respectively. Define the long-range order parameter L as $L=\left(\sum_{i=1}^{N}\sigma_i\right)/N=(N_+-N_-)/N$. The system is placed under an external magnetic field B.

(1) Show that the Hamiltonian can be written as

$$-J\sum_{n.n.}\sigma_i\sigma_j - B\mu\sum_i \sigma_i = -J\left(4N_{++}-qNL-\frac{1}{2}qN\right)-B\mu NL .$$

(2) The Bragg-Williams approximation ignores the short-range order by assuming $N_{++}/(qN/2)\approx(N_+/N)^2$. Comment on why the $N_{++}/(qN/2)\approx(N_+/N)^2$ ignores short-range order and show that the average energy U under the Bragg-Williams approximation is given by

$$U=-\frac{1}{2}qJN\overline{L}^2-\mu BN\overline{L}$$

where the bar means the average value.

SM-5

Consider a non-interacting Fermi gas in a large 3-dimensional volume V, with non-relativistic single-particle dispersion law $\varepsilon(p)=\frac{p^2}{2m}$.

(1) Show that the grand-canonical partition function can be expressed as

$$\mathcal{Q}=\prod_p\left(1+e^{-\beta(\varepsilon_p-\mu)}\right).$$

(2) Derive an expression for the total particle number N, identifying the average occupation number $\langle n_p\rangle$ in the limits of high and low temperatures.

(3) Given $PV=k_BT\ln\mathcal{Q}$, show that $PV=\frac{2}{3}E$, where P is the pressure and E is the total energy.

(4) Compute the first three coefficients (c_0, c_1, c_2) of the virial expansion for the pressure, i.e. expressing P in the expansion in the powers of the fugacity $z=e^{\beta\mu}$ (c_0 is the coefficient for the term of z^0, c_1 for z^1, and c_2 for z^2). Note that $\ln(1+x)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$ for $|x|<1$.

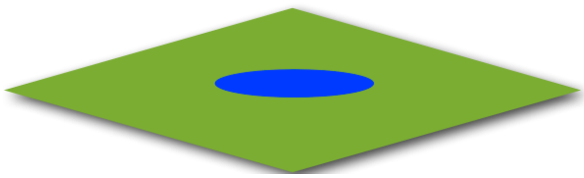
EMI-1

1. Prove Green's theorem for the electrostatic potential, namely

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}')dv' + \frac{1}{4\pi}\left[\int_{\partial V} G(\mathbf{r},\mathbf{r}')\nabla'\Phi(\mathbf{r}')\cdot d\mathbf{S}' - \int_{\partial V}\Phi(\mathbf{r}')\nabla' G(\mathbf{r},\mathbf{r}')\cdot d\mathbf{S}'\right] \tag{1}$$

where V is a closed volume bounded by a surface ∂V, and $G(\mathbf{r},\mathbf{r}')$ is an appropriately chosen, normalized Green's function.

2. A flat, infinite conducting surface is held at a constant electrostatic potential V_0. The surface has a circular hole, which is filled with a circular conductor held at a different constant potential V_1. This second conductor is separated from the rest of the surface by some thin layer of insulating material. Using the previous result, find a surface-integral expression for the electrostatic potential in the region above the plane.



3. Consider the same problem as before, but now the plane is made of some material such that the electrostatic potential is held at a constant normal derivative $\partial\Phi/\partial n = W_0$, and the circular area is held at constant normal derivative W_1. Find a surface-integral expression for the electrostatic potential in the region above the plane.

EMI2

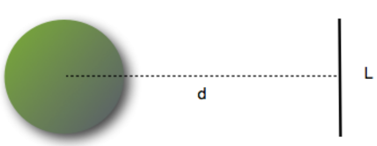
1. Starting from Maxwell's equations, show that in the absence of external currents $\mathbf{J}$, the magnetic field $\mathbf{H}$ is determined by a scalar magnetostatic potential Φ_M. Find the equation obeyed by this potential in the presence of ferromagnetic materials.

2. A classical model due to London for type I superconductors (those where the magnetic field $\mathbf{B}$ vanishes inside the sample) assumes that there exists a constitutive relation between a stationary current $\mathbf{J}$ and the vector potential $\mathbf{A}$, of the form

$$\mathbf{J} = -\frac{1}{\mu_0\lambda_L^2}\mathbf{A}. \tag{2}$$

Using the above relation and Maxwell's equations, find the equation obeyed by $\mathbf{B}$ and show that λ_L is a penetration length scale for any static $\mathbf{B}$ inside the superconductor.

3. Now assume that there are no currents and $\mathrm{B} \equiv \mathbf{0}$ inside a spherical piece of superconducting material of radius R. A thin magnet of length L and constant axial magnetization density $\mathbf{M}$ is set in front of it. The center of the magnet lies at a distance d from the center of the sphere. The ends of the magnet are equidistant from the center of the sphere (see figure).

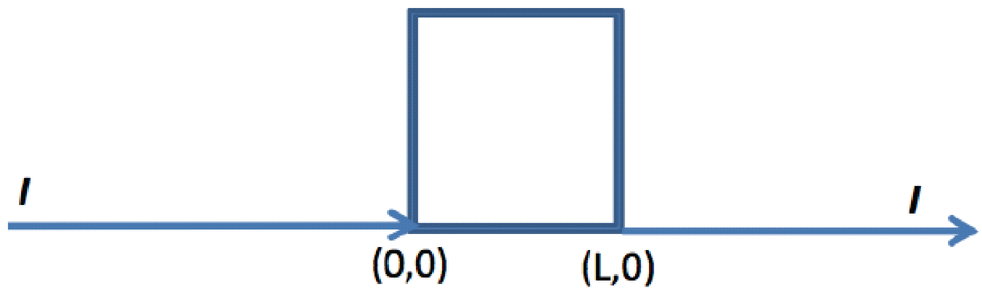


Write down the magnetization of the magnet as a function of the coordinates (for some axes of your choice), using appropriate distributions (Dirac delta functions and Heaviside theta functions).

4. Find the magnetostatic potential outside the magnet. You may assume that the magnetization density $\mathbf{M}$ is unaffected by the sphere.

EMI-3

Consider a uniform wire loop shaped as a square of side length L, placed on the xy plane. A semi-infinite straight wire is connected to one corner, another is connected to the next corner, as shown in the figure. The current I flows from left to right.



1. Taking into account that the wire is uniform, find the fraction of the current I, flowing through each of the sides of the square.

2. Using the above result, find the direction and magnitude of the magnetic induction (B) field at the center of the square.

EMI-4

An electric dipole moment p is placed at the center of a spherical cavity of radius R in a grounded conductor. The (total) potential (for $r \leq R$) receives contributions from the dipole as well as induced charge on the cavity wall.

1. Find the potential inside the cavity due to the induced charge.

2. Find the potential for all points inside the cavity.

3. Calculate the surface charge density induced on the cavity wall.

EMI-5

Consider a magnetic charge g located at the origin so that the magnetic field satisfies

$$\nabla \cdot \mathbf{B} = 4\pi\delta(\mathbf{r}) \tag{3}$$

Away from the origin, the magnetic field is divergenceless, so one may expect it to be the curl of a vector potential $\mathbf{A}$:

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{4}$$

1. Show that this cannot be true everywhere by considering the volume integration of $\nabla \cdot \mathbf{B}$ around the origin where the magnetic charge is located. It turns out that $\mathbf{B} = \nabla \times \mathbf{A}$ fails to hold on a line (called Dirac string), which we may take to be the positive z axis:

$$\mathbf{B} = \nabla \times \mathbf{A} + C\delta(x)\delta(y)\Theta(z)\mathbf{e}_3 \tag{5}$$

where $\mathbf{e}_3$ is a unit vector on the +z axis, and $\Theta(z)$ is the step function (being 1 for positive z, and 0 for negative z). Find the constant C.

2. Consider a magnetic charge g located at the origin with a particle with electric charge e and mass m moving around it. What is the equation of motion of the particle $m\frac{d\mathbf{v}}{dt} = ...$?

3. Show that the associated moment equation

$$\mathbf{r} \times m\frac{d\mathbf{v}}{dt} = ... \tag{6}$$

can be simplified to read

$$\frac{d\mathbf{J}}{dt} = 0, \tag{7}$$

where

$$\mathbf{J} = \mathbf{r} \times m\mathbf{v} - \frac{eg}{c}\frac{\mathbf{r}}{r} \tag{8}$$

is, hence, a constant of motion, and can be identified as the total angular momentum operator.

4. Assume that all the components of $\mathbf{J}$ are quantized in units of $\hbar$. Show that $eg/c = n\hbar$, with n being an integer or half-integer. This is the Dirac condition for the quantization of electric charge.

QMI-1

Vector operators:

(a) Write an expression for $[J_i, A_j]$, the commutator of a Cartesian component of the angular-momentum operator $\vec{J}$ with a Cartesian component of another arbitrary vector operator $\vec{A}$.

(b) Use the result to show that the dot product $\vec{A} \cdot \vec{B}$ of two vector operators is a scalar operator.

(c) Show that $\vec{A} \times \vec{B}$ is a vector operator.

QMI-2

(a) Write down the quantization condition for bound-state energies in one dimension in the WKB approximation.

(b) Use the approximation to find the (approximate) bound-state energies for the one-dimensional potential $V(x) = \alpha x^4$.

Hint:

$$\int_0^1 dy\,\sqrt{1-y^4} = \frac{\Gamma(1/4)\Gamma(3/2)}{4\Gamma(7/4)} = 0.8740\ldots$$

(c) Use the approximation to find an algebraic equation for the (approximate) bound-state energies in the potential

$$V(x) = \begin{cases} k(x-a) & x \geq a \\ 0 & 0 < x < a \\ V(-x) & x < 0 \end{cases},$$

with $a, k > 0$.

QMI-3

A particle with mass m and electric charge q is trapped in a one dimensional harmonic oscillator potential:

$$V(x) = \frac{1}{2}m\omega^2 x^2$$

It also experiences an external, uniform electrical field of strength E_0 applied along the $+x$ direction. For $t < 0$ the particle is in the ground state of this hamiltonian. At $t = 0$, the electric field is abruptly turn off. Find the probability that a particle will be in the nth energy eigenstate the instant after the field is turned off. The nth eigenstate for the SHO is given as:

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} H_n(\xi) e^{-\xi^2/2}$$

with $\xi = \sqrt{\frac{m\omega}{\hbar}}x$. You may also find the generating function identity of Hermite Polynomials useful for this problem:

$$e^{-z^2+2z\xi} = \sum_{n=0}^{\infty} \frac{z^n}{n!} H_n(\xi)$$

QMI-4

A particle of mass m is constrained to move freely on only the curved surface of a spherical hemisphere (half of a sphere).

(a) Find an expression for the energy-levels of the particle in this system and justify your answer.

(b) Write down, in terms of spherical harmonics, the wave functions and degeneracies for the first three energy levels.

You may assume without proof the following properties of Associated Legendere functions for this problem:

$$P_l^m(-x) = (-1)^{l+m} P_l^m(x)$$

If $P_l^m(x)$ has even parity, then $P_l^m(0) \neq 0$

QMI-5

Consider a spin-1/2 particle in a uniform magnetic field along z direction. The Hamiltonian is given by $H = \omega S_z$. At time $t = 0$ the state is $|\alpha\rangle = a\,|+\rangle + b\,|-\rangle$, where a and b are real numbers.

(a) Calculate the expectation values $\langle S_x\rangle_0 , \langle S_y\rangle_0$ at $t = 0$.

(b) Show that at a late time t,

$$\langle S_x\rangle_t = \langle S_x\rangle_0 \cos\omega t - \langle S_y\rangle_0 \sin\omega t$$
$$\langle S_y\rangle_t = \langle S_y\rangle_0 \cos\omega t + \langle S_x\rangle_0 \sin\omega t$$

EMII-1

Consider Maxwell's equations in the absence of material media and sources.

1. Write down Maxwell's equations and show that they can be written in manifestly Lorentz-covariant form. What is the physical significance of this result?

2. Prove that if the electric and magnetic fields are orthogonal in a given inertial reference frame, they are orthogonal in all inertial frames.

3. Show that if a given inertial frame observes a pure electric field, there is no inertial frame in which the field is purely magnetic.

EMII-2

Consider Maxwell's equations without sources inside an infinite cylindrical cavity of arbitrary cross section.

1. Write down Maxwell's equations and show that, if the cavity is filled with a uniform non-dissipative medium of permittivity ϵ and permeability μ, then Maxwell's equations imply

$$\left(\nabla^2 + \mu\epsilon\omega^2\right) \mathrm{E} = 0 \tag{1}$$

$$\left(\nabla^2 + \mu\epsilon\omega^2\right) \mathrm{B} = 0 \tag{2}$$

   where a harmonic time dependence with frequency ω is assumed for the fields.

2. By separating variables in the longitudinal and transverse directions, show that

$$\left(\nabla_t^2 + \mu\epsilon\omega^2 - k^2\right) \mathrm{E} = 0 \tag{3}$$

$$\left(\nabla_t^2 + \mu\epsilon\omega^2 - k^2\right) \mathrm{B} = 0, \tag{4}$$

   where k is an unspecified integration constant, and $\nabla_t^2 = \nabla^2 - \frac{\partial^2}{\partial z^2}$ is the transverse part of the Laplacian operator.

3. Re-write Maxwell's equations to show that the transverse components of the fields, E_t and B_t are completely determined by the longitudinal components E_z and B_z.

EMII-3

An electron and a positron (mass m_e) with equal Lorentz factors γ approach each other with a relative angle of θ between their directions of motion (assume their directions of motion are at angles $\pm\theta/2$ relative to the x axis). The pair annihilates and creates two photons. One photon has energy E_1 and moves forward along the x axis, while the second photon has energy E_2 and moves in the opposite direction.

1. In terms of the provided parameters, determine the energies of the photons, E_1 and E_2.

2. In the limit as $\theta \to 0$, determine the way in which the two photon energies depend upon γ (for $\gamma \gg 1$).

EMII-4

A dextrose solution is optically active and is characterized by a constitutive relation for its polarization of the form

$$\mathbf{P} = \gamma\, \nabla \times \mathbf{E} \tag{5}$$

where γ is a real constant that depends only on the dextrose concentration, which is assumed to be homogeneous. The solution is non-conducting and non-magnetic.

1. Write down Maxwell's equations for such a dextrose solution. Derive the equation obeyed by the electric field.

2. Consider a plane electromagnetic wave of (real) angular frequency ω propagating in the z direction. Write down the form of the electric field $\mathbf{E}$ of such a plane wave and, using the previous result, find the possible propagation modes for such a wave.

EMII-5

1. Start with the infinitesimal space-time interval (invariant under Lorentz transformation). Show that the proper time $d\tau$ of a particle in terms of the time interval dt in the coordinate system in which it is moving with an instantaneous velocity $\vec{v}$ is given by $d\tau = dt/\gamma$ where $\gamma = (1-\beta^2)^{-1/2}$ and $\vec{\beta} = \vec{v}/c$. Then derive the relativistic 4-velocity of a particle moving with velocity $\vec{v}$.

2. Start with the Larmor formula for the power of radiation of a slowly moving charged particle (with charge e) $P = \frac{2e^2}{3c^3}(d\vec{v}/dt)^2$. Show that the power of radiation for a fast moving charged particle in a circle with radius ρ is approximately given by $P = \frac{2e^2 c}{3\rho^2}\beta^4\gamma^4$.

3. Show that the power of radiation of a fast moving charged particle in general is given by $P = \frac{2e^2}{3c^3}(\vec{\alpha}^2 - \alpha_0^2)$ where $\vec{\alpha} = \gamma^4\vec{\beta}\cdot\dot{\vec{v}}\vec{\beta} + \gamma^2\dot{\vec{v}}$, $\alpha_0 = -\gamma^4\vec{\beta}\cdot\dot{\vec{v}}$ and $\dot{\vec{v}} = d\vec{v}/dt$.

QMII-1

Consider a spinless particle of mass m that experiences a three-dimensional hamiltonian:

$$H = \frac{p^2}{2m} + k_1 x^2 + k_2 y^2$$

where k_1 and k_2 are real-valued constants. You may ignore translational motion along the z-axis for this problem. Show explicitly for this system that the angular momentum along the z-axis is generally not conserved and find a relationship that must k_1 and k_2 satisfy for it to be conserved.

QMII-2

<u>Problem 2</u>

Consider a two level system with $\mathbf{E_1 < E_2}$. There is a time dependent potential that connects the two levels as follows:

$$\mathbf{V_{11} = V_{22} = 0, \qquad V_{12} = \gamma\ e^{\,i\omega t},\quad V_{21} = \gamma\ e^{-i\omega t}} \qquad (\gamma \text{ is real})$$

At $\mathbf{t = 0}$ it is known that only the lower level is populated, i.e. $\mathbf{c_1(0) = 1}$ and $\mathbf{c_2(0)=0}$.

a) Find $\mathbf{|c_1(t)|^2}$ and $\mathbf{|c_2(t)|^2}$ for $\mathbf{t>0}$ by *exactly* solving the coupled differential equation

$$\mathbf{i\hbar\,(dc_k / dt) = \Sigma^{2}_{n=1}\ V_{kn}(t)\, e^{(i\omega_{kn} t)}\, c_n \qquad (k=1,2)}$$

b) Do the same problem using time dependent perturbation theory to lowest nonvanishing order. Compare the two approaches for small value of $\boldsymbol{\gamma}$. Treat the following two cases separately i) ω very different from ω_{21} and, ii) ω very close to ω_{21}.

(Fact: These are known as Rabi's formula)

QMII-3

Suppose, we want to solve the problem of a particle in a potential $V(r) = -Ae^{-r/a}$ which is a model for the binding energy of a deuteron due to the strong nuclear force, with A=32 MeV and a=2.2 fm. The strong nuclear force does not exactly have this form, since unlike in the case of the Coulomb interaction we don't know what the exact form should be, but the above potential is a reasonable model.

1.Estimate the ground state energy by choosing a trial wave function for the ground state and compare it to the exact answer E = -2.245 MeV.

2. Describe the symmetry considerations you used for choosing the trial wave function.

QMII-4

1. A particle of momentum K is scattered off a spherical symmetric scattering potential $U(r) = U_0 \exp(-r^2/R^2)$.

(a) Using the first order Born approximation to show that the differential cross section is $\frac{d\sigma}{d\Omega} = u^2 R^2 \exp\left(-\frac{q^2R^2}{2}\right)$, where $q = 2K\sin\frac{\theta}{2}$ is the momentum transfer, θ is the scattering angle, and $u = \frac{\sqrt{\pi} m U_0 R^2}{2\hbar^2}$.

(b) Given the differential cross section of (a) show that the total scattering cross section is $\sigma = \frac{2\pi u^2}{K^2}(1 - \exp(-2K^2R^2))$.

QMII-5

2-electron system

1.a) Consider the total spin angular momentum operator of two electrons, $\vec{S}_{total} = \vec{S}_1 \oplus \vec{S}_2$. Using the standard representation for each electron, $\vec{S}_1 = \frac{\hbar}{2}\vec{\sigma}$, with σ_3 spin eigenstates $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$, find the eigenvalues of $\vec{S}^2_{Total}$ for the spin singlet and spin triplet states of the two-electron system. Show the derivation.

b.) Describe the total wave function of the two electron system as the product of a spatial wave function $\psi(\vec{r}_1, \vec{r}_2)$ and the spin states discussed in (a). From your understanding of the statistics of electrons, what is the symmetry property under the exchange of the two electrons of the spatial wave function and of the spin wave function in each of the following:

the para state

the ortho state.

c.) If the two electrons discussed above are part of a multi-electron atom, describe which of the two states in (b) is lower in energy and why. Explain why an anti-symmetric spatial wave function implies a lower probability for small separations of the electrons than a symmetric one.

AstroI-1

1. a. Use dimensional arguments and the equation of state for a non-relativistic degenerate gas to derive a mass-radius relation for white dwarfs.

b. Two white dwarfs have the same temperature, but one is brighter. Which is the more massive?

c. Use dimensional arguments to show that the pressure and density go to zero as the mass goes to zero. Clearly, the density can't be zero, but what you've shown is actually correct. Explain this apparent paradox.

AstroI-2

2. Show that the stellar luminosity L and mass M have the following approximate scaling relations:

a. For very massive stars ($M >> 30M_{\odot}$), $L \propto M$. Thompson scattering is the main source of opacity in these stars and $P_{gas} << P_{rad}$.

b. For intermediate to high mass stars $5M_{\odot} << M << 30M_{\odot}$, $L \propto M^3$. Again, Thompson scattering is the main source of opacity, but $P_{gas} >> P_{rad}$.

c. Low mass stars ($M \approx 1M_{\odot}$), $L \propto M^b$, where $b \approx 5.5$. However, these stars are more complicated. Here $P_{gas} >> P_{rad}$ and the opacity is dominated by free-free transitions with $k \propto \rho T^{-7/2}$. In this case you'll also need an approximate mass-radius relation. Assume that these stars burn hydrogen via the pp chains with $\epsilon \propto \rho T^4$.

AstroI-3

3. a. Near the solar photosphere, the Rosseland mean opacity is $k_R \approx 0.24$ cm^2 /g, which is mainly due to bound-free and free-free absorption of trace amount of H^-). Given that the opacity of H^- is $\approx 7.2 \times 10^{-32} \rho(\mathrm{g/cm^3})^{1/2} T(\mathrm{K})^9$ cm^2/g, What is the density of the photosphere?

b. What is the ionization fraction of H atoms at the solar photosphere? What is the electron number density? (Hint: In doing the calculation, you can assume that gas consists only of H atoms, free electrons and protons; The abundances of other species are much smaller).

c. The H atom can capture an electron to form H^- and release 0.75 eV (i.e., the binding energy of H^- is 0.75 eV). What is the H^-/H ratio in the solar photosphere?

Hint: You'll need to calculate partition functions to do parts b) and c). For the H atom in it's ground state, there are two spin orientations for the electron - spin up and spin down. If excited states can be neglected, then $U_H = 2$. You can use this logic to calculate the partition functions for H^+ and H^-. However, you'll need to argue that you can neglect excited states in calculating U.

AstroI-4

4. a. A star of mass M uses up all of its nuclear fuel. Find its rate of contraction if the luminosity L = const. Show that for times much greater than some characteristic time, the rate of contraction is

$$\dot{r} \approx \frac{-\alpha G M^2}{2Lt^2}, \tag{1}$$

where α is a constant determined by the distribution of matter within the star.

b. Show that when a self-gravitating body of polytropic gas shrinks homologously and adiabatically, its thermal energy scales with radius R as $E_{th} \propto R^{3(1-\gamma)}$. Show that a polytropic star is unstable to gravitational collapse if $\gamma \leq 4/3$.

AstroI-5

5. A 1000 K spherical gray body source of radius 1 m and emssivity $\epsilon_\lambda = 0.9$ is viewed in air by a detector distant 1 km. The entrance aperture of the optics is a circle of radius 5 mm, has field of view diameter 0.2°, and transmits 90% of the incident light to the detector. The detector operates at 1 μm wavelength and has 0.01 μm bandpass.

a. Compute the luminosity of the sphere and the power received by the detector.

b. Estimate the average number of photons detected each second.

Speed of light	c	$2.99792458 \times 10^{10}$ cm/s
Gravitational constant	G	6.67259×10^{-8} cm^3/g/ s^2
Planck constant	h	$6.6260755 \times 10^{-27}$ erg s
Boltzmann constant	k	1.380658×10^{-16} erg/K
		8.617385×10^{-5} eV/K
Elementary charge	e	$4.8032068 \times 10^{-10}$ esu
Atomic mass unit	u	1.660540×10^{-24} g
		931.5 MeV/c^2
Mass of electron	m_e	$9.1093897 \times 10^{-28}$ g
		0.511 MeV/c^2
Mass of proton	m_p	$1.6726231 \times 10^{-24}$ g
		938.3 MeV/c^2
Mass of neutron	m_n	$1.6749286 \times 10^{-24}$ g
		939.6 MeV/c^2
Mass of ^{1}H atom	m_H	$1.6735344 \times 10^{-24}$ g
Stefan–Boltzmann constant	σ	5.67051×10^{-5} erg/cm^2/K^4/s
Thomson scattering constant	σ_T	6.6524×10^{-25} cm^2

Solar mass	$M_\odot$	1.9891×10^{33} g
Solar radius	$R_\odot$	6.95508×10^{10} cm
Solar luminosity	$L_\odot$	3.8458×10^{33} erg/s
Solar effective temperature	$T_{\mathrm{eff}\,\odot}$	5777 K
Earth's mass	$M_\oplus$	5.9742×10^{27} g
Earth's radius	$R_\oplus$	6.378136×10^{8} cm
Astronomical unit	AU	$1.4959787066 \times 10^{13}$ cm
Light-year	ly	$9.460730472 \times 10^{17}$ cm
Parsec	pc	3.0856776×10^{18} cm
		3.26167 ly

AstroII-1

Assume that an E0 elliptical galaxy's mass density distribution is spherically symmetric and given by a Jaffe model,

$$\rho(r) = \frac{M}{4\pi} \frac{a}{r^2(r+a)^2},$$

where M is the total mass and a is a characteristic radius.

1. Determine the mass $M(r)$ contained within a radius r and confirm that the parameter M is the total mass.
2. Calculate the gravitational potential $\Phi(r)$ (subject to the condition $\Phi(\infty) = 0$).
3. Determine the radial component of the gravitational acceleration $a_r(r)$.
4. Obtain the profile $v_c(r)$ for circular orbit velocities as a function of r.
5. Find the leading part of the asymptotic expansion of $\Phi(r)$ for $r/a \to \infty$. Find the leading part of the expansion for $r/a \to 0$. Comment on the two results.

AstroII-2

Consider an axisymmetric galactic potential $\Phi(R, z)$, where R, z, and φ are cylindrical coordinates. Consider a star of unit mass ($m = 1$) moving in this potential, so that $\Phi(R, z)$ is the potential energy and

$$T = \frac{1}{2}\left[\dot{R}^2 + \dot{z}^2 + (R\dot{\varphi})^2\right],$$

is the kinetic energy.

1. Write down the effective potential $\Phi_{\text{eff}}(R, z)$ for two-dimensional motion in R and z with constant J_z.
2. Assume the potential is symmetric across the equatorial plane, $z = 0$ (i.e., $\Phi(R, z) = \Phi(R, -z)$). What are the conditions on the effective potential and energy E such that $R = R_0$ and $z = 0$ represent a circular orbit moving in φ with constant velocity Ω?
3. Assume the effective potential can be expanded to have the form

$$\Phi_{\text{eff}}(R, z) = \Phi_{\text{eff}}(R_0, 0) + \frac{1}{2}\kappa^2(R - R_0)^2 + \frac{1}{2}\nu^2 z^2 + \cdots,$$

   referred to as the epicyclic approximation. Find the equations of motion in R and z, give the form of $R(t)$ and $z(t)$, and sketch the motion in the R-z plane.

4. What are the three constants of the motion in the case of an infinitely thin disk? Why is the total angular momentum J not a constant of the motion?

AstroII-3

Consider a spherical gas cloud of uniform temperature T and density ρ.

a) For the range of sizes for which the pressure in the gas cloud can be considered negligible compared to the collapse time, write down the collapse time in terms of ρ.

b) Assuming the sound speed is related to the temperature by $v_s = \sqrt{\frac{3kT}{2m}}$ where m is the mean particle mass, estimate the critical cloud size at which the pressureless assumption breaks down and the cloud is stable against collapse.

c) By computing the self-potential energy and kinetic energy of the cloud, prove that the critical cloud size in part b is roughly equivalent to the virial radius for a cloud of this temperature and density.

d) Suppose that the cloud is located in a rotating galaxy disk. Explain qualitatively how the critical cloud size for stability will change and why. Since this change implies that the stable cloud size will no longer be approximately equal to the virial radius computed above, has the virial theorem been violated?

AstroII-4

Consider a group of 40 galaxies with radius ~1 Mpc and typical orbital velocity ~300 km/s.

a) Recall that the change in velocity for galaxy 1 caused by a distant weak encounter with galaxy 2 is $\Delta V_\perp = \frac{2Gm_2}{bV}$ where b is the impact parameter and V is their relative velocity. Roughly at what minimum impact parameter b_{min} does the weak encounter approximation break down? For a perturbing galaxy of mass ~10^{10} M_{sun}, estimate the value of b_{min}. Comment on the scale of b_{min} compared to typical galaxy diameters and separations.

b) Approximating the group as having constant density and the perturbing galaxy mass as $m_2 \sim 10^{10}$ M_{sun} on average, prove that the mean square random velocity $\langle \Delta V_\perp^2 \rangle$ that develops for galaxy 1 perpendicular to V due to many weak galaxy encounters in time t is $\langle \Delta V_\perp^2 \rangle = \frac{8\pi G^2 m_2^2 nt}{V} \ln\left(\frac{b_{max}}{b_{min}}\right)$.

c) From the result in part b, what is the two-body relaxation time t_{relax} as a function of V, m_2, and n?

d) Using conservation of kinetic energy, explain qualitatively how the increase in $\langle \Delta V_\perp^2 \rangle$ leads to dynamical friction and thus to the merger of the group galaxies. From the V dependence of t_{relax}, explain why this process is quicker in groups than in clusters.

AstroII-5

Consider a thin galaxy disk, roughly 4 kpc thick and 40 kpc in diameter.

a) Use Gauss's theorem (the divergence theorem) to show that just outside the disk, the restoring force is $2\pi G\Sigma$ toward the plane of the disk (i.e., $F_z = -2\pi G\Sigma \frac{|z|}{z}$), where Σ=Σ(r) is the surface mass density of the disk. Why is this calculation correct only just outside the disk and not to arbitrarily large |z|?

b) Suppose that the density is nearly constant $\rho=\rho_c$ at small z-heights within the disk, |z|<100pc. In this regime, expand the potential as a Taylor series to second order in z.

c) Prove that the vertical force at |z|<100pc is $F_z=-4\pi G\rho_c z$. What type of motion occurs at these small z-heights? How do you know it does not continue to arbitrarily large |z|?

<u>Possibly Useful Formulae</u>

G = 4.3 x 10^{-6} kpc $(\mathrm{km/s})^2/M_{sun}$

1kpc ≈ 1Gyr x 1km/s

Poisson Equation $\nabla^2\Phi = -4\pi G\rho$

spherically symmetric case $\nabla^2\Phi \rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right)$

thin disk case $\frac{\partial^2\Phi}{\partial z^2} = -4\pi G\rho$

Wien's Law T = 3 mm·K / λ_{peak}

tidal acceleration $a=2Gm(2r)/d^3$ where d = the distance between galaxies, m = the mass of the perturbing galaxy, and r = the radius of the galaxy being perturbed

dynamical friction $\frac{dV}{dt} = -\frac{M\rho}{V^2}$

ram pressure of hot gas $P_{ram}=\rho_{hot} V_{rel}^2$

freefall time $t_{ff} = \sqrt{\frac{3\pi}{32G\rho}}$

dynamical time $t_{dyn} = \sqrt{\frac{3\pi}{16G\rho}}$

crossing time t_{cross} = R/V

relaxation time $t_{relax} \sim \frac{0.1N}{\ln N} t_{cross}$

dynamical friction decay time $t_{df} = \frac{2.64\times10^{11}}{\ln\Lambda}\left(\frac{r_i}{2kpc}\right)^2\left(\frac{v_c}{250km/s}\right)\left(\frac{10^6 M_{sun}}{M}\right)yr$

Useful facts:
$F = Gm_1m_2/d^2$
$G=4.3 \times 10^{-6}$ kpc (km/s)^2/M_{sun}
1kpc = 1Gyr x 1km/s
Virial Theorem 2KE+PE=0
Poisson Equation $\nabla^2\varphi = -4\pi G\rho$
Wien's Law T = 3 mm·K / λ_{peak}
centripetal force $F=mV^2/r$ (uniform circular motion)
Faber-Jackson Relation $L \propto \sigma^4$
Dynamical time $t_{dyn}=\sqrt{2}t_{ff}$
Crossing time t_{cross} = R/V = 1 Gyr (**R** in kpc/**V** in km/s)
Relaxation time t_{relax} = 0.1N / ln(N) t_{cross} = 10^6 yr x 0.1N/ln(N) x (**R** in pc/**V** in km/s)

Numerical Constants:

Solar Mass (Msun):	1.989×10^{33} g
Solar Radius (Rsun):	6.96×10^{10} cm
Solar Luminosity:	3.847×10^{33} erg/s
Gravitational Constant (G)	6.6726×10^{-8} cm^3/g/s^2
Proton mass	1.6726×10^{-24} g = 938.27 MeV/c^2
Yield of p-p reactions (Q)	26.7 MeV = 4.28×10^{-5} ergs
Boltzmann constant	1.38×10^{-16} erg/K
Planck's Constant	6.626×10^{-27} erg-s
Electron mass	9.109×10^{-28} g

$m_p = 1.6726 \times 10^{-24}$ g