UNIVERSITY of NORTH CAROLINA at CHAPEL HILL

Doctoral Written Examination in Physics, 2012

Part I: Classical mechanics and Statistical mechanics

Saturday, May 12, 2012

Instructions: Please work in the assigned room, but take a break outside anytime you want to. Mathematical handbooks and electronic calculators are allowed.
Begin each answer on a new sheet and write only on one side of each sheet.
Identify each sheet by:

Page___________ of Question___________Student's # (PID)_________________

CM: Classical Mechanics
Work out 3 out of 5 problems

SM: Statistical Mechanics
Work out 3 out of 5 problems

(Partial credit will be given for partial answers)

My work is completed in full observance of the Honor code:

Signature_________________________

Print name____________________________

UNIVERSITY of NORTH CAROLINA at CHAPEL HILL

Doctoral Written Examination in Physics, 2012

Part II: Electromagnetism I and Quantum mechanics I

Monday, May 14, 2012

Instructions: Please work in the assigned room, but take a break outside anytime you want to. Mathematical handbooks and electronic calculators are allowed.
Begin each answer on a new sheet and write only on one side of each sheet.
Identify each sheet by:

Page___________ of Question___________Student's # (PID)_________________

EMI: Electromagnetism I
Work out 3 out of 5 problems

QMI: Quantum Mechanics I
Work out 3 out of 5 problems

(Partial credit will be given for partial answers)

My work is completed in full observance of the Honor code:

Signature_________________________

Print name____________________________

UNIVERSITY of NORTH CAROLINA at CHAPEL HILL

Doctoral Written Examination in Physics, 2012

Part III: Electromagnetism II and Quantum mechanics II

Monday, May 14, 2012

Instructions: Please work in the assigned room, but take a break outside anytime you want to. Mathematical handbooks and electronic calculators are allowed.
Begin each answer on a new sheet and write only on one side of each sheet.
Identify each sheet by:

Page___________ of Question___________Student's # (PID)_________________

EMII: Electromagnetism II
Work out 3 out of 5 problems

SM: Quantum mechanics II
Work out 3 out of 5 problems

(Partial credit will be given for partial answers)

My work is completed in full observance of the Honor code:

Signature_________________________

Print name____________________________

UNIVERSITY of NORTH CAROLINA at CHAPEL HILL

Doctoral Written Examination in Physics, 2012

Part III: Astro I and II

Monday, May 14, 2012

Instructions: Please work in the assigned room, but take a break outside anytime you want to. Mathematical handbooks and electronic calculators are allowed.
Begin each answer on a new sheet and write only on one side of each sheet.
Identify each sheet by:

Page___________ of Question___________Student's # (PID)_________________

Astro I:
Work out 3 out of 5 problems

Astro II:
Work out 3 out of 5 problems

(Partial credit will be given for partial answers)

My work is completed in full observance of the Honor code:

Signature________________________

Print name___________________________

CM-1

A point particle of mass m moves in a plane in response to the following Lagrangian

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + 2\alpha\dot{x}\dot{y}\right) - \frac{1}{2}k\left(x^2 + y^2 + 2\beta xy\right),$$

where $k > 0$ is a spring constant and α and β are two other time-independent parameters.

(a) Find the normal mode frequencies, $\omega_{1,2}$.

(b) What conditions must be placed on α and β so that the motion is always a bounded, stable oscillation?

(c) Find the eigenvectors of the system.

(d) Sketch the motion in the x and y coordinate system for the two eignemodes. How does the behavior depend upon the values of α and β?

CM-2

Consider a straight cylindrical shaft of radius $\alpha \approx 1\,\mathrm{m}$ that penetrates through the center of the earth. It emerges at latitudes θ and $-\theta$ relative to the equator. Assume that the earth is a perfectly homogenous sphere of mass M and radius R that rotates with angular velocity ω. A test point mass of mass m is suspended at rest relative to the shaft directly above the symmetry axis of the shaft and then released to free-fall down the shaft. You may neglect air-friction and relativistic effects for this problem. Answer the following questions:

(a) Use a non-inertial Cartesian coordinate system where one axis points down the center of the shaft and another along the east-west direction to provide a *qualitative* description of the test mass' motion.

(b) Find the equations of motion of the test mass in the reference frame from part (a). Recall that the fictitious force experienced by a mass m in a reference frame rotating at a constant angular velocity is given by:

$$\mathbf{F} = \mathbf{F}_{\text{Coriolis}} + \mathbf{F}_{\text{centrifugal}} = -2m\mathbf{\Omega} \times \mathbf{v_r} - m\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r})$$

(c) Discuss the motion in the specific cases where $\theta = 0°$ and $\theta = 90°$. If the test mass experiences any oscillatory motion, find the period of the oscillation(s).

(d) In the "shallow approximation", α is small enough so that the test mass only falls a short distance, $d \ll R$, before it hits the side of the shaft. Find an algebraic equation for d in the shallow approximation. Do not attempt to solve it.

(e) Assume that the test mass is dropped down the hole in the northern hemisphere. If you stand over the shaft so that north is at 0 radians, at what angle will the test mass hit the side of the shaft in the shallow approximation?

CM-3

A particle of mass m moves under the influence of gravity, with downward acceleration g, on the inner surface of the paraboloid of revolution, $x^2+y^2 = az$. Here $a > 0$ is a length scale and the surface is assumed to be frictionless. The coordinate z points upward in the vertical direction.

(a) Write down the Lagrangian and show that the equations of motion can have the form

$$\ddot{\rho} - \rho\dot{\phi}^2 = 2\lambda\rho,$$
$$d(\rho^2\dot{\phi})/dt = 0,$$
$$\ddot{z} = -g - a\,\lambda,$$

and

$$a\,z = \rho^2,$$

where we have converted to cylindrical coordinates (ρ, ϕ, z).

(b) Now let the particle be given an angular velocity of $\sqrt{2g/a}$, so that it orbits in a horizontal circle. Prove the stability of the particle in this circular path by showing that if the particle is displaced slightly from this path, while holding the angular momentum fixed, it will undergo oscillations about the path.

(c) Find the frequency of the radial oscillations.

CM-4

Consider the Lagrangian for a two-dimensional system

$$L = \dot{q}_1\dot{q}_2 - cq_1q_2\,,$$

where c is a positive constant.

(a) Solve the equations of motion and describe the physical system that the Lagrangian defines.

(b) The Lagrangian is invariant under the scale transformation

$$q1 \longrightarrow e^{\lambda}q_1\,, \qquad q_2 \longrightarrow e^{-\lambda}q_2\,,$$

for arbitrary λ. Use Noether's theorem to find the conserved quantity associated with this invariance, and interpret its meaning.

(c) Suppose the two coordinates q_1 and q_2 are the x and y coordinates of a single object. What property of the object does the conserved quantity describe?

CM-5

A ball is bouncing vertically and perfectly elastically in an elevator that accelerates from rest with acceleration $a(t)$. The rate of change of the acceleration $a(t)$ is very slow: $\dot{a}(t)T \ll g$, where T is the period of the ball's motion and g is the usual acceleration of objects in the earth's gravitational field.

(a) The equivalence principle says that this situation is equivalent to what other situation?

(b) If the ball has maximum height h_0 above the floor of the elevator before the acceleration begins, what is the maximum height $h(t)$ at a later time t, in the adiabatic limit?

SM-1

Consider an ideal Bose gas (non-relativistic) confined to a region of area A in two dimensions. Express the number of particles in the excited states, N_ε, and the number of particles in the ground state, N_0, in terms of z, T, and A, and show that the system does not exhibit Bose-Einstein condensation unless $T \to 0$ K.

($g_n(z) = \frac{1}{\Gamma(n)} \int_0^\infty \frac{x^{n-1}dx}{z^{-1}e^x - 1}$; $0 \le z \le 1$ and $g_n(1) = \zeta(n)$ and $\zeta(1) = \infty$)

SM-2

Consider N identical, localized, noninteracting spins with spin quantum number j. The magnetic moment of each spin is $\vec{\mu} = g\mu_B \vec{j}$ where μ_B is the Bohr magneton, g is the Lande factor, and the eigenvalues of j_z, the magnetic quantum number m, are $m = -j, -j+1, \ldots j-1, j$. In the presence of an external magnetic field $\vec{H}$ the energy of the system is given by $E = -\sum_{i=1}^{N} \vec{\mu}_i \cdot \vec{H}$.

(1) Show that the magnetization is given by $M_z = Ng\mu_B j B_j(x)$

where $x = g\mu_B Hj / k_B T$ and $B_j(x)$ is the Brillouin function given by

$$B_j(x) = (1+\frac{1}{2j}) \coth\left((1+\frac{1}{2j})x \right) - \frac{1}{2j} \coth\left(\frac{x}{2j} \right)$$

(2) Show that for $x = g\mu_B Hj / k_B T \ll 1$, the magnetic susceptibility is given by $\chi = N\frac{g^2 \mu_B^2 j(j+1)}{3k_B T}$

($\coth x \approx x^{-1} + \frac{1}{3}x$ for $x \ll 1$)

SM-3

Consider a classical gas of N identical particles. The energy of the system is given by

$$H = \sum_{i=1}^{N} \frac{\vec{p}_i^{\,2}}{2m} + \sum_{i<k} U_{ik}(|\vec{r}_i - \vec{r}_k|)$$

In the dilute (atomic volume $\times N \ll V/N$) and high temperature ($|U| \ll k_B T$) approximation it can be shown that the partition function can be written as

$$Z(T,V,N) = \frac{1}{N!}\left(\frac{1}{\lambda^3}\right)^N Q_N(V,T) \quad \text{where} \quad \lambda = \frac{h}{\sqrt{2\pi m k_B T}}$$

where the configurational integral $Q_N(V,T)$ is given by

$$Q_N(V,T) = V^N + V^{N-2}\sum_{i<k}\int d^3r_i \int d^3r_k \left(e^{-U_{ik}/k_BT} - 1\right)$$

Assume the potential is given by the hard sphere potential

$$U_{ik}(|\vec{r}_i - \vec{r}_k|) = \begin{matrix} +\infty & |\vec{r}_i - \vec{r}_k| < r_0 \\ 0 & |\vec{r}_i - \vec{r}_k| \geq r_0 \end{matrix}$$

Show that the equation of state is given by

$$P\left(V - N\frac{2\pi}{3}r_0^3\right) = Nk_BT$$

SM-4

Consider the closed (this means that the nearest neighbors of spin 1 are spin 2 and spin N) 1 D Ising model where the Hamiltonian is given by

$$H_N(\sigma_1,\ldots,\sigma_N) = -I\sum_{n.n.}\sigma_i\sigma_k - \mu B\sum_{i=1}^{N}\sigma_i \ ; \quad \sigma_i = \pm 1 \text{ and } n.n.\text{=nearest neighbors}$$

μ is the magnetic moment, B is the magnetic field, and I is the coupling strength. The mean-field approximation predicts a spontaneous magnetization below a critical temperature $T_c = 2I/k_B$. The exact solution shows that the mean-field approximation made qualitatively wrong prediction. Such phase transition does not exist in 1D. Show that the exact free energy is given by

$$F = -k_BT\ln\left(\lambda_1^N + \lambda_2^N\right)$$

where

$$\lambda_{1,2} = e^{I/k_BT}\cosh\left(\mu B/k_BT\right) \pm \sqrt{e^{-2I/k_BT} + e^{2I/k_BT}\sinh^2\left(\mu B/k_BT\right)}$$

From this you can derive the magnetization and see why there is no spontaneous magnetization at T>0 K. However, you don't have to show this here.

SM-5

A cylinder of radius R and length L contains N molecules of mass m of an ideal gas at temperature T. The cylinder rotates about its axis with an angular velocity ω. Find a change in the free energy of the gas ΔF, as compared to that at rest.

EMI.1

(a) For an arbitrarily moving charge, the charge and current densities are $\rho(\vec{r},t) = e\delta(\vec{r}-\vec{R}(t)), \vec{j}(\vec{r},t) = e(d\vec{R}/dt)\delta(\vec{r}-\vec{R}(t))$, where $\vec{R}(t)$ is the position of the charged particle. Verify the statement of conservation of charge.

(b) Find the total charge and the electric dipole moment of the charge density $\rho(\vec{r}) = -\vec{d}\cdot\nabla\delta(\vec{r})$.

(c) What electromagnetic fields do the following potentials describe? $\phi = 0, \vec{A} = \vec{a}(\vec{a}\cdot\vec{r})$, where $\vec{a}$ is a constant vector.

EMI.2

Start with the potential due to a given local charge distribution around the origin of the coordinate system $\phi(\vec{r}) = \int(d\vec{r'})\frac{\rho(\vec{r'})}{|\vec{r}-\vec{r'}|}$. If the total charge of the given charge distribution is zero, show that (a) the potential , in its leading behavior for large distances, has the form
$\phi(\vec{r}) = \frac{\vec{r}\cdot\vec{d}}{r^3}$ where the electric dipole moment is given by $\vec{d} = \int(d\vec{r'})\vec{r'}\rho(\vec{r'})$.

(b) Consider an additional point charge e_1 located at a point $\vec{r}$ lying far from the dipole; the interaction energy is given by $E = \vec{d}\cdot\frac{e_1\vec{r}}{r^3}$. Show that we can interpret this energy as the interaction energy of the dipole moment with the electric field $\vec{E}$ produced by e_1 at the origin, i.e., $E = -\vec{d}\cdot\vec{E}$.

(c) Use $E = -\vec{d_1}\cdot\vec{E}$ as the interaction energy of an electric diple moment $\vec{d_1}$ with the field $\vec{E}$ produced by a given charge distribution far from $\vec{d_1}$. Calculate the interaction energy E for dipole-dople interaction, i.e., for the interaction of $\vec{d_1}$ at the origin with the field $\vec{E}$ produced by another dipole moment $\vec{d_2}$ located at $\vec{r}$.

EMI.3

(a) Show that a perfectly conducting sphere of radius a placed in a constant magnetic field $\vec{B}_0$ acquires a magnetic moment $\vec{\mu} = -\frac{1}{2}a^3\vec{B}_0$.

(b) Find the surface current density $\vec{K}$.

(c) Show that the values for $\vec{\mu}$ and $\vec{K}$ are consistent with each other.

EMI-4

Consider two straight parallel wires, carrying static charge with linear charge density of ρ and $-\rho$, respectively. The wires are along the z-direction, one is located at $x = {}^{a}/_{2}$ and the other at $x = -{}^{a}/_{2}$.
(a) Find the electric potential and the electric field everywhere in space.
(b) Simplify your expression for the region far away from the wire, and express the field in terms of the linear dipole density $p=\rho a$.

EMI-5

Consider two parallel plane electrodes (regarded as infinite) separated by a distance d. The cathode located at $x = 0$ with electric potential of $\varphi(x = 0) = 0$ is capable if emitting unlimited electrons (charge *e* and mass *m*) when an electric field is applied to it. The electrons leaving the cathode with zero initial velocity are accelerated toward the anode located at $x = d$ with electric potential of $(x = d) = V_0$. In the steady state there will be a constant electric current flowing from the cathode to the anode.
(a) Find a relationship between the current density J, the space charge density $\rho(x)$ and the electric potential $\varphi(x)$ in the space between the two electrodes. Is *J* a constant or a function of *x*, *why?*
(b) Derive a differential equation that determines the electric potential $\varphi(x)$.
(c) Assuming a power law solution ($\varphi(x)$ is proportional to x^p), solve for the potential density *J* in terms of *e*, *m*, *d*, and V_0.

QM1-1 Two electrons interact via a spin-spin interaction that is given as $\alpha\, \mathbf{S_1} \cdot \mathbf{S_2}$ where α is a constant. *One* of the electrons is also trapped in a region with a homogeneous external magnetic field of intensity B_0. Please answer the following questions:

(a) What is the sign of α? Explain your answer.

(b) Consider *only* spin degrees of freedom and find the allowed energies of this system in terms of fundamental constants and α.

(c) Assume that we create an ensemble of these two electron systems. For each member of this ensemble, we perform a measurement of the spin of the electron that is trapped in the magnetic field region along the direction of the magnetic field. What is the average energy of the subset of the ensemble with measured spin in the same direction as the magnetic field?

QM I-2

(a) Consider a Hamiltonian $H(\lambda)$ that depends on a parameter λ, one of the Hamiltonian's eigenstates $|\varphi(\lambda)\rangle$, and the corresponding energy $E(\lambda)$. Show that

$$\frac{dE(\lambda)}{d\lambda} = \langle\varphi(\lambda)|\, \frac{dH(\lambda)}{d\lambda}\, |\varphi(\lambda)\rangle \ ,$$

a result that is known as the Hellman-Feynman theorem.

(b) The states of the hydrogen atom with no radial nodes $(n = l + 1)$ have energies

$$E(l) = \frac{-e^4 m}{2\hbar^2 (l+1)^2}$$

Letting $H(\lambda)$ be the Hamiltonian for the radial Schrödinger equation and the parameter λ be l, use the Hellman-Feynman theorem to derive an expression for the expectation value of the operator r^{-2} in states with no radial nodes.

QM1-2 Consider an electron that is free to jump between 3 fixed, identical atoms in a molecule. Each atom is located at a corner of an equilateral triangle. Ignore spin and any other nearby electrons and atoms. We can define an orthornormal basis set of states of the electron to be spherically symmetrical orbitals bound to each atom. In other words, $|S_i\rangle$ would correspond to an electron bound to the ith atom. In this basis the Hamiltonian for the system has all off-diagonal elements equal to ϵ and the diagonal elements equal to zero. Please answer the following questions:

(a) Given that one of the energy eigenvalues is 2ϵ, find the other energy eigenvalues.

(b) The Hamiltonian is obviously invariant under rotations of $2\pi/3$. Find a matrix to represent such a rotation operator (R) and find the simultaneous eigenstates of H and R. You may find the fact that $R^3 = 1$ useful.

(d) Assume that at $t = 0$ the electron is in the $|S_1\rangle$ state. Find the probability that the electron will stay bound to that electron as a function of time.

QMI-4

A paradox:

(a) Show that for finite-dimensional matrices A and B,

$$\mathrm{Tr}[A, B] = 0\,.$$

(b) In the 1-d harmonic oscillator, the raising and lowering operators $a^\dagger$ and a obey the commutation relation

$$\left[a, a^\dagger\right] = I\,.$$

The trace of I is obviously not zero. In the basis of oscillator eigenstates, write down the matrix representations of a and $a^\dagger$ (you can write down the upper left parts and indicate the rest with dots). Multiply them together to get the matrix representations of $aa^\dagger$ and $a^\dagger a$ and explain why the result from part a) doesn't apply.

QMI-5

(a) Write down or derive the equation of motion for the density operator $\hat{\rho}(t)$.

(b) Use the solution of (a) to show whether or not a mixed state can evolve into a pure state.

(c) The *reduced density operator* for a two-particle system is defined as the trace of the two-paticle density operator over the states of the second particle. For the simple two-particle density operator $\rho = |\alpha_1\alpha_2\rangle\langle\beta_1\beta_2|$, the reduced density operator is given by

$$\rho_R = |\alpha_1\rangle\langle\beta_1| \ \mathrm{Tr}\left[|\alpha_2\rangle\langle\beta_2|\right] .$$

All two-particle density operators can be written as sums of simple ones like that above, and ρ_R is defined for these more complicated cases in the obvious way, by invoking linearity.

The reduced density operator is an effective operator for a single particle that takes into account our complete ignorance of the other particle. Consider a two-spin density operator associated with the pure spin-singlet state, i.e.

$$\rho = |S=0\rangle\langle S=0|$$

where

$$|S=0\rangle = \frac{1}{\sqrt{2}}\left(|{+}{-}\rangle - |{-}{+}\rangle\right) .$$

Find the reduced density operator. Does it correspond to a pure single-particle state or a mixture of single-particle states? If the former, what state, if the latter, what polarization?

EMII-1

A nonrelativistic particle of mass m and charge e, and initial kinetic energy E, makes a head-on collision with a fixed central force region with potential energy $V(r)$. The particle comes from an infinite distance away. The potential energy steadily increases toward the center so that

$$V(r) < E, \qquad \text{for} \qquad r > r_0,$$

$$V(r) > E, \qquad \text{for} \qquad r < r_0.$$

1. Find the instantaneous total radiated power as a function of position.
2. Integrate the power over all time to find an expression for the total radiated energy.
3. Assuming the potential energy and its derivative at the turning point are finite, show that the integrated emission is finite.

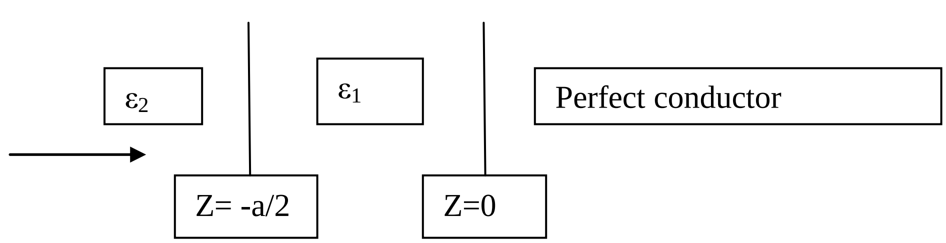
EMII-2

A high energy photon of energy E encounters an electron of mass m and charge e at rest. A scattering occurs with a photon of energy E' emerging and moving at an angle θ relative to the direction of motion of the original photon. The electron recoils with some Lorentz factor γ and angle ψ.

1. Find the expression for the scattered photon's energy in terms of the original photon's energy E, the electron mass m, and the scattering angle θ.
2. Derive an expression for the Lorentz factor $\gamma = E_e^{\mathrm{recoil}}/(mc^2)$ of the recoiling electron.

EMII-3

Two parallel dielectric media are backed by a perfect electric conductor as shown in the accompanying figure. A source, to the left of the first interface, initiates an incident plane wave described by $e^{ik_2(z+a/2)}$ with perpendicular polarization (i.e., the electric field being perpendicular to the plane of incidence.) The reflected wave is represented in terms of $r_a e^{-ik_2(z+a/2)}$. Find the reflection coefficient r_a. Next, find the absolute value of the coefficient and then give a brief physical interpretation of the result.



EMII-4

Start with the Maxwell's equations in vacuum in terms of the electric field $\vec{E}$, the magnetic field $\vec{B}$, the charge density ρ and the current density $\vec{j}$. Write $\vec{E}$ and $\vec{B}$ in terms of scalar potential ϕ and vector potential $\vec{A}$.

(a) Show that, in the Lorenz gauge, the potentials obey the differential equations: $-D\vec{A} = 4\pi\vec{j}/c$ and a similar equation for ϕ, where D stands for the d'Alembertian.

(b) Show that the potentials can be solved in terms of the sources $\vec{j}$ and ρ by using the Green's function technique with the Green's function obeying the equation $-DG(\vec{r}-\vec{r'}, t-t') = 4\pi\delta(\vec{r}-\vec{r'})\delta(t-t')$.

(c) Assume that the equation has been solved (you do NOT have to solve the equation) to yield $G(\vec{r}-\vec{r'}, t-t') = \frac{1}{|\vec{r}-\vec{r'}|}\delta(\pm\frac{1}{c}|\vec{r}-\vec{r'}| - (t-t'))$. Which sign should we use and why? Finally write down the potentials as integral equations of the sources.

EMII-5

Consider a coaxial waveguide. Let the inner radius be b, the outer radius a. Assume $a - b << a$. Find the cutoff wavenumbers for the TM mode. [Hint: You do not need Bessel functions and you may note that for $\Phi(\rho)$ satisfying $(\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho} - \frac{m^2}{\rho^2} + \gamma^2)\Phi(\rho) = 0$, one has $(\frac{d^2}{d\rho^2} - \frac{m^2}{\rho^2} + \frac{1}{4}\frac{1}{\rho^2} + \gamma^2)\sqrt{\rho}\Phi(\rho) = 0$.]

QMII-1

a) Assuming that the hamiltonian is invariant under time-reversal, prove that the wavefunction for a spinless non-degenerate system at any given instant of time can always be chosen to be real.
b) The wavefunction for a plane wave state at t=0 is given by a complex function $\mathbf{e}^{\mathbf{i\,px}}$. Why does this not violate time reversal invariance?

QMII-2

A p-orbital electron characterized by $|n, l=1, m = +1, -1, 0\rangle$ (ignore spin) is subjected to a potential $\mathbf{V = \lambda (x^2 - y^2)}$ where λ =constant.

a) Obtain the 'correct' zero-order energy eigenstates that diagonalize the perturbation. You don't need to evaluate the energy shifts in detail, but show that the original 3-fold degeneracy is now completely removed.
b) Because V is invariant under time reversal and because there is no longer any degeneracy, we expect each of the energy eigenstates obtained in (a) to go into itself, (up to a phase factor) under time-reversal. Check this point explicitly.

QMII-3

Work out the quadratic Zeeman effect for the ground state of hydrogen atom, $[\langle x|0\rangle = 1/(\pi a_0^3)^{1/2}\, e^{-r/a_0}]$ due to the neglected term " $\mathbf{e^2 A^2 / 2m_e c^2}$ " in the hamiltonian taken to first order. Write the energy shift as $\Delta = -\chi B^2/2$ and obtain an expression for " χ ".

This is a useful integral: $\int_0^\infty e^{-ar} r^n \, dr = n!/(a^{n+1})$.

QMII-4

Three spin -0 particles are situated at the corners of an equilateral triangle. Let us define the z-axis to go through the center and in the direction normal to the plane of the triangle. The whole system is free to rotate about the z-axis. Using statistics considerations, obtain restrictions on the magnetic quantum numbers corresponding to $\mathbf{J_z}$.

QMII-5

Consider scattering from the delta-shell potential $V(r) = g\,\delta(r - r_0)$.

a) First determine the boundary conditions at $r = 0$ and $r = r_0$, then make a suitable ansatz, apply the boundary conditions, and compute the s-wave scattering amplitude.
b) Determine the s-wave bound states of an infinite spherical well of radius r_0. Comment on the relation of the delta –barrier resonance and these bound states. What happens to the s-wave scattering length when the incident *k*-value sweeps across the "*k*" corresponding to one of these quasi bound state

AstroI-1 White Dwarfs

a) The pressure integral: $$P = \frac{1}{3}\int_0^\infty pvn(p)dp$$

allows you to calculate pressure given a distribution in momentum *n(p)dp*. Assuming that a completely degenerate electron gas has the electrons packed as tightly as possible, so that their separation is of order $n_e^{-1/3}$, use the Heisenberg uncertainty principle to estimate the momentum of an electron in terms of n_e. By further assuming that $p = m_e v$ (non-relativistivic) and that all the electrons have the same momentum (to make the integral trivial), derive the exponent η in the power law equation of state:

$$P \propto \rho^\eta T^0$$

(For this problem, don't worry about the constants of proportionality, they'll be wrong under the constant momentum assumption anyway.)

b) Now use your understanding of this equation of state and hydrostatic equilibrium and mass conservation in scaling law form to plot white dwarf cooling curves on a log Teff,- log L (H-R) diagram. Work in solar units and use the normalization that an 0.6 solar mass white dwarf has a radius of 0.01 Rsun at solar Teff. Plot curves for 0.2, 0.6, 0.8 1md 1.0 solar mass white dwarfs.

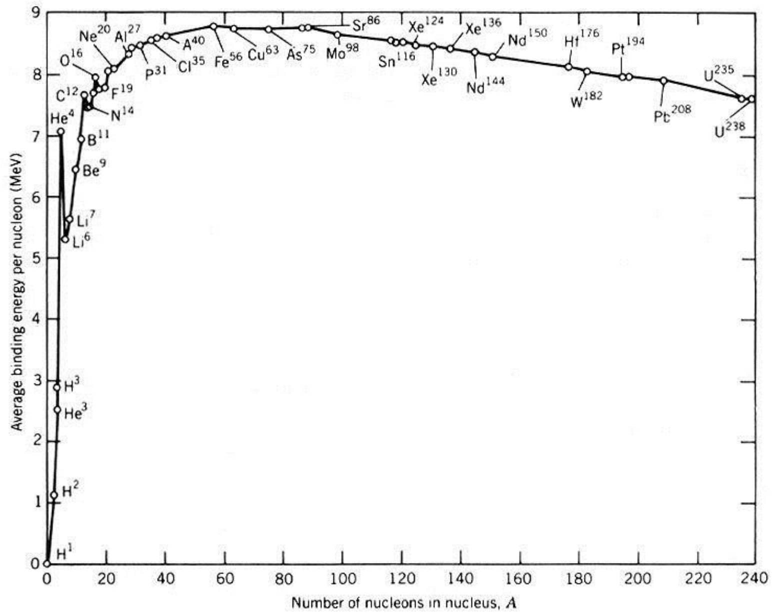
c) Now write the mass-luminosity relationship for non-relativistic white dwarfs in power law form.

AstroI-2 Deuterium burning in stars

a) In the formation of a main sequence star from a protostar there is a phase in which primordial Deuterium is fused. This happens at a temperature of 10^6 degrees rather than the 15 x 10^6 required for P-P reactions. Use hydrostatic equilibrium and mass conservation, along with the ideal gas law, to compare the radius of a 1 solar mass protostar in its D burning phase to its radius on the main sequence. You need to assume the density profiles of the two stars are identical (one can be scaled to the other).

b) Referring to the curve of binding energy below, estimate the total energy available from D burning for a solar mass star if the primordial D abundance is 0.013% of P, and occurs in the inner 10% of the star. (Deuterium fuses via 1H + 2H –> 3He + γ)

c) Compare the D-burning timescale to the Kelvin Helmholtz (gravitational contraction) timescale.

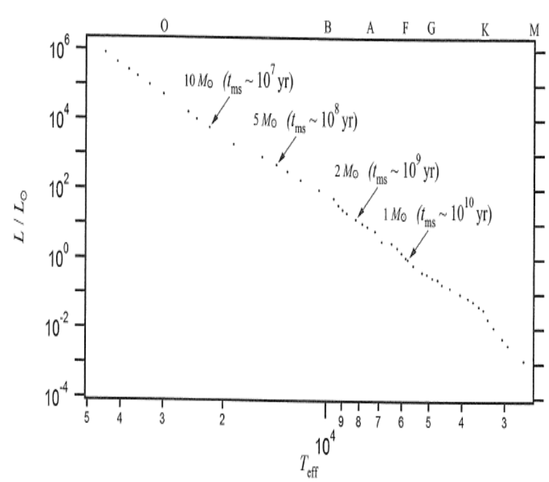


AstroI-3 Lifetime, luminosity, mass scaling

a) Use the plot below to estimate the exponent for a power law relation for main sequence lifetime in terms of stellar mass: $L \propto M^{\alpha}$

b) Assuming that all main sequence stars convert the same fraction of their total mass to He, what is the expected Mass-Luminosity relation on the main sequence? How closely do the luminosities expected from this relation match the luminosities seen in the H-R diagram depicted? Comment.

c) Assuming the gas in these stars is ideal (a fairly good assumption) and that the central temperature is proportional to the effective temperature (not so good), use the assumption of hydrostatic equilibrium and the Mass-Luminosity relation from above to estimate the temperature dependence of the nuclear reactions (assume negligible density dependence).



Astro I-4

α Centauri A is G2 like the Sun but older, having 1.14 $M_\odot$, 1.23 $R_\odot$, and 1.5 $L_\odot$. It is 1.34 parsecs from us in an 80-year elliptical orbit with α Centauri B. B is type K1, has 0.92 $M_\odot$, 0.86 $R_\odot$, 0.5 $L_\odot$, and approaches A to 11.2 AU separation. Assume that both stars emit as blackbodies and consider a planet in a circular orbit around B.

(a) Assume that the planet cools as a blackbody through a non-greenhouse gas atmosphere that reaches 1 Earth surface pressure. Calculate the inner (water steam formation) orbital radius in AU of the "habitable" zone around B, ignoring for now star A. Take planet albedo as 50% and assume rapid rotation.

(b) Show quantitatively that there is no significant change in the zone's outer radius (water ice formation) around B even when star A is closest.

(c) Like Earthlings, the α Centaurians are loading CO_2 into their atmosphere as they rapidly burn up fossil fuels. Assuming that their planet has average temperature 40 °C but that the increasing CO_2 will soon push it to 50 °C, calculate

  i. what fraction of the sunlight should be blocked by a perfectly opaque sunscreen to reduce insolation on the planet to restore its pre-CO_2 equilibrium temperature, or alternatively,

  ii. by how much they must increase planetary albedo at visible wavelengths through chemical modification.

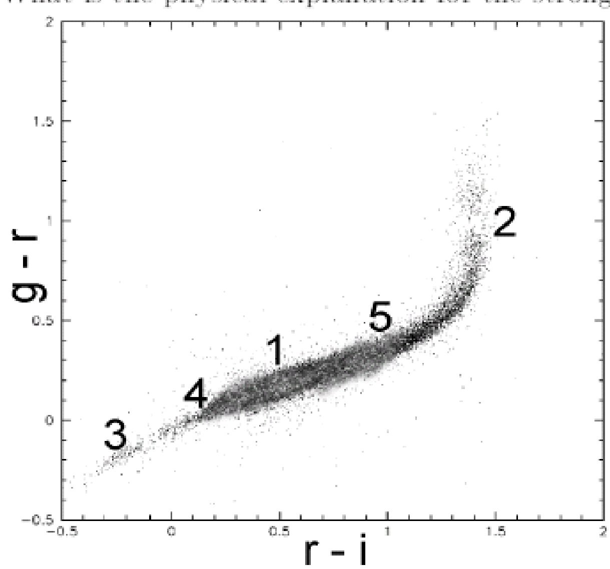
AstroI-5

The diagram below plots colors of many stars with Sloan r < 17 through the 4 SDSS filters indicated, over a field in our Galaxy. Assume that the colors have been corrected for reddening.

(a) Where feasible, identify the spectral classes, evolutionary stages, and approximate masses of stars near each of the numbered regions.

(b) Assuming that most of these stars formed in a burst, estimate the mean age today of this population if they show similar chemical composition to that of the Sun.

(c) What is the physical explanation for the strongly curved "tail" at 2?



AstroII-1

It is possible to calculate the nuclear statistical equilibrium at high densities between neutrons, protons, and electrons in a neutron star by treating each as an ideal Fermi-Dirac gas component. For equilibrium to occur there has to be a balance between

$$n \to p + e^- + \bar{\nu}_e,$$

and

$$p + e^- \to n + \nu_e,$$

and in a neutron star we assume the neutrinos escape.

At high enough densities the muon (another fermion) can appear, changing an ideal $n - p - e$ gas into an ideal $n - p - e - \mu$ gas. If it is energetically feasible, the two reactions,

$$\mu^- \to e^- + \nu_\mu + \bar{\nu}_e,$$

and

$$e^- \to \mu^- + \bar{\nu}_\mu + \nu_e,$$

may occur.

(a) Write down the set of thermodynamic equilibrium equations between dimensionless Fermi momenta x_e, x_p, x_n, and x_μ, and the expression for the total energy density.

(b) In terms of the masses of the particles, m_e, m_p, m_n, and m_μ, determine x_e, x_p, and x_n at threshold for the appearance of muons in the gas. (You do not need to obtain numerical values.)

AstroII-2

The Universe contains cosmic ray particles, including a very high energy power-law distribution. The very highest energy cosmic ray protons are measured to have energies up to around 10^{20} eV. At that energy the spectrum cuts off, though there is controversy over the statistics of the very highest energy events.

The GZK mechanism is thought to limit the highest energy that a proton can have because scattering of the relativistic proton off of a cosmic microwave background photon can, at a certain threshold, produce a pion. Once threshold for pion production is reached, the proton loses approximately 20% of its energy per pion scattering. The reaction is

$$p + \gamma_{\rm cmb} \rightarrow p' + \pi^0,$$

where the most favorable case is for the CMB photon to be traveling in the opposite direction of the initial proton.

(a) Let the proton mass be m_p, the pion mass be m_π, and the CMB photon energy be E. The Lorentz factor of the proton before scattering is γ and after scattering is γ'. Assume the reaction is just at threshold to make a pion. Find the expression for the required initial proton Lorentz factor γ in terms of the other masses and energies.

(b) Take the proton mass to be $m_p = 938$ MeV, the pion mass to be $m_\pi = 135$ MeV, and the CMB photon energy to be $E = 2.5 \times 10^{-4}$ eV (note: eV). Find the approximate GZK cutoff energy for the protons.

ASTR II Problem 3

A thin accretion disk surrounds a Schwarzschild black hole. The gas can be treated as if approximately in isolated circular orbits. Recall that (more general) radial orbital motion satisfies

$$\left(\frac{dr}{d\tau}\right)^2 = \tilde{E}^2 - V(r),$$

where the effective potential $V(r)$ is

$$V(r) = \left(1 - \frac{2M}{r}\right)\left(1 + \frac{\tilde{L}^2}{r^2}\right).$$

Here $\tilde{E}$ is the relativistic specific energy ($\tilde{E} \to 1$ for particles just unbound at infinity) and $\tilde{L}$ is the specific angular momentum (units with $G = c = 1$).

(a) Go through the calculation and show that the innermost stable circular orbit (ISCO) is at $r = 6M$.

(b) Find the values of $\tilde{E}$ and $\tilde{L}$ at the ISCO.

Assume that $\xi = 1 - \tilde{E}$ is the radiative efficiency of the disk. Assume further that at any given time the disk is being fed with mass at just the right rate $\dot{M}$ to maintain Eddington luminosity, $dE/dt = L_{\rm edd}$.

(c) Material at the ISCO plunges into the black hole via a short spiral. Assume the ISCO orbital constants are preserved during this brief plunge. Derive an equation for the growth of the black hole mass (in terms of the various physical constants **including** G and c). Solve for $M(t)$ assuming that $M = M_0$ at $t = 0$.

(d) Estimate how long it takes for the black hole to spin up to $a/M = J/M^2 \simeq 0.9$.

In parts (c) and (d) continue to treat the black hole as if it remains a Schwarzschild black hole.

ASTROII-4. Binary Survival in Supernova Explosion and Kick Velocity

a. A progenitor of a supernova of mass M_p and a companion star of mass M_c are in a circular orbit about each other of semi-major axis $a = a_p + a_c$, where a_p and a_c are with respect to the system's center of mass. Determine expressions for a_p/a and a_c/a.

b. Determine an expression for the angular speed $\omega_p = \omega_c = \omega$ of the stars an a function of $M = M_p + M_c$ and a.

c. Determine expressions for the velocities v_p, v_c, and $v = v_p + v_c$ of the stars as functions of M_p, M_c, M, and a.

d. The progenitor supernovas leaving behind a neutron star of mass M_{NS}. Determine an expression for the minimum mass that the companion star must have for the binary to survive (as a function of M_p and M_{NS}).

e. Determine an expression for the momentum of the supernova shell.

f. Determine an expression for the kick velocity of the binary system.

ASTROII-5. Colonization of the Galaxy

Assuming that humanity has mastered efficient, controlled fusion as an energy source, estimate how long it will take for us to colonize the Galaxy. (The text is long, but the calculations are short.)

a. Assume that the mass of our unfueled ships is similar to the mass of our fuel supply (^{3}He, collected from gas giants at each stop). Also assume that we are generating energy via ^{3}He + ^{3}He $\rightarrow$ ^{4}He + 2p and converting it to kinetic energy near 100% efficiency (difficult to do, but theoretically possible). Very roughly, estimate how fast our ships would go. (The simplest back-of-the-envelope estimate matches the exact calculation within a factor of $\sim$2, so do not waste time on the exact calculation.)

$m_p = 1.007276$ u

$m_{^3He} = 3.016029$ u

$m_{^4He} = 4.002602$ u

b. How long would it take our ships to travel from our location to the far side of the Galaxy if not interrupted by stopping to colonize worlds?

c. Roughly, how many star systems are in the Galaxy? Roughly, what is the volume of the Galaxy in cubic light years? Consequently, what is the typical distance between star systems? How long would it take our ships to travel this distance?

d. Assume that all stars have at least one planet in the traditional habitable zone. But also assume that humanity is not interested in tidally-locked planets. If the atmosphere is thin, only the ring around the planet in constant twilight would be habitable (a far way to travel for not much surface area). If the atmosphere is thick, it will redistribute the heat from the star-facing side to the dark side and the entire surface would be habitable, but the winds could be violent.

The distance at which planets tidally lock to their stars scales with $M^{1/3}$, where M is the mass of the star. In our solar system, Mercury is tidally locked but Venus is not. The distance at which planets are in the traditional habitable zone scales with $L^{1/2}$, where L is the luminosity of the star. In our solar system, Earth is in the middle of the traditional habitable zone. Assuming a reasonable stellar mass-luminosity relation, estimate below what stellar mass M roughly Earth-mass/size planets near the middle of the traditional habitable zone are tidally locked.

e. Very roughly, what fraction of stars have masses above M? Also very roughly, of these what fraction of stars are not in binary (or multiple) systems? (In binary systems, planets can only be in stable orbits if close enough to one of the stars to be tidally locked, or if far enough from both stars to be outside of the traditional habitable zone.) Given these factors, and any others that you wish to include, estimate what fraction of stars have planets acceptable for colonization (assuming that the remaining stars have at least one planet, or moon around a hot Jupiter, in the traditional habitable zone that can be sufficiently terraformed; we will not consider worlds in non-traditional habitable zones, such as tidally-heated moons of regular, cold Jupiters).

f. Given your estimate, what is the typical distance between habitable planets? How long would it take our ships to travel this distance?

g. Generational ships must be large enough to support a population with sufficient genetic diversity, but not so large of a population as to make the ship too expensive/time-consuming to build and fuel. Assume 100 - 1000 people per ship. Given humanity's current level of technological and medical advancement, our population is doubling every 40 years. Arguments can be made for both faster and slower growth rates, with future levels of technological and medical advancement, but slowed by the challenges of terraforming and/or (presumably controlled) genetic adaptation to the new environment. Simply using humanity's current rate, roughly estimate how long it would take a generational crew to fully populate an Earth-like planet?

h. Assume that we fully populate each world before we send out new generational ships, presumably in all (unsettled) directions. The time that it would take to colonize the entire Galaxy is then no different than the time that it would take to colonize an approximately direct path from our location to the far side of the Galaxy (of course avoiding the Galactic center). Given your estimates, how long is this time?

Is this timescale short or long? Specifically, if this timescale is much shorter than the typical timescale for a habitable planet to develop life and civilization to the point of mastering fusion, the first civilization to master fusion will likely get the whole Galaxy. Given that we are probably only hundreds of years away from mastering fusion (at most), in this case the Galaxy is either 100% ours, or it has already been fully colonized by an earlier civilization which for whatever reason decided to leave us alone (but in which case the Galaxy is 0% ours). If this timescale is much longer than the typical timescale for a habitable planet to develop life and civilization to the point of mastering fusion, the Galaxy will likely be colonized by many civilizations, in which case we are likely to get at least a part of it, but only a part. Assuming that humanity survives the next few hundred years, what does our long-term future look like?

Useful facts:

$F = Gm_1m_2/d^2$

$G=4.3 \times 10^{-6}$ kpc (km/s)^2/M_{sun}

1kpc = 1Gyr x 1km/s

Virial Theorem 2KE+PE=0

Poisson Equation $\nabla^2\varphi = -4\pi G\rho$

Wien's Law T = 3 mm·K / λ_{peak}

centripetal force $F=mV^2/r$ (uniform circular motion)

Faber-Jackson Relation $L \propto \sigma^4$

Dynamical time $t_{dyn}=\sqrt{2}t_{ff}$

Crossing time t_{cross} = R/V = 1 Gyr (**R** in kpc/**V** in km/s)

Relaxation time t_{relax} = 0.1N / ln(N) t_{cross} = 10^6 yr x 0.1N/ln(N) x (**R** in pc/**V** in km/s)

Numerical Constants:

Solar Mass (Msun):	1.989×10^{33} g
Solar Radius (Rsun):	6.96×10^{10} cm
Solar Luminosity:	3.847×10^{33} erg/s
Gravitational Constant (G)	6.6726×10^{-8} cm^3/g/s^2
Proton mass	1.6726×10^{-24} g = 938.27 MeV/c^2
Yield of p-p reactions (Q)	26.7 MeV = 4.28×10^{-5} ergs
Boltzmann constant	1.38×10^{-16} erg/K
Planck's Constant	6.626×10^{-27} erg-s
Electron mass	9.109×10^{-28} g

$m_p = 1.6726 \times 10^{-24}$ g