UNIVERSITY of NORTH CAROLINA at CHAPEL HILL

Doctoral Written Examination in Physics, 2011

Part I: Classical mechanics and Statistical mechanics

Friday, May 7, 2011

Instructions: Please work in the assigned room, but take a break outside anytime you want to. Mathematical handbooks and electronic calculators are allowed.
Begin each answer on a new sheet and write only on one side of each sheet.
Identify each sheet by:

Page___________ of Question___________Student's # (PID)_________________

CM: Classical Mechanics
Work out 3 out of 5 problems

SM: Statistical Mechanics
Work out 3 out of 5 problems

(Partial credit will be given for partial answers)

My work is completed in full observance of the Honor code:

Signature_________________________

Print name____________________________

UNIVERSITY of NORTH CAROLINA at CHAPEL HILL

Doctoral Written Examination in Physics, 2011

Part II: Electromagnetism I and Quantum mechanics I

Monday, May 9, 2011

Instructions: Please work in the assigned room, but take a break outside anytime you want to. Mathematical handbooks and electronic calculators are allowed.
Begin each answer on a new sheet and write only on one side of each sheet.
Identify each sheet by:

Page___________ of Question___________Student's # (PID)_________________

EMI: Electromagnetism I
Work out 3 out of 5 problems

QMI: Quantum Mechanics I
Work out 3 out of 5 problems

(Partial credit will be given for partial answers)

My work is completed in full observance of the Honor code:

Signature________________________

Print name___________________________

UNIVERSITY of NORTH CAROLINA at CHAPEL HILL

Doctoral Written Examination in Physics, 2011

Part III: Electromagnetism II and Quantum mechanics II

Monday, May 9, 2011

Instructions: Please work in the assigned room, but take a break outside anytime you want to. Mathematical handbooks and electronic calculators are allowed.
Begin each answer on a new sheet and write only on one side of each sheet.
Identify each sheet by:

Page___________ of Question___________Student's # (PID)_________________

EMII: Electromagnetism II
Work out 3 out of 5 problems

SM: Quantum mechanics II
Work out 3 out of 5 problems

(Partial credit will be given for partial answers)

My work is completed in full observance of the Honor code:

Signature_________________________

Print name____________________________

UNIVERSITY of NORTH CAROLINA at CHAPEL HILL

Doctoral Written Examination in Physics, 2011

Part III: Astro I and II

Monday, May 9, 2011

Instructions: Please work in the assigned room, but take a break outside anytime you want to. Mathematical handbooks and electronic calculators are allowed.
Begin each answer on a new sheet and write <u>only</u> on <u>one side</u> of each sheet.
Identify each sheet by:

Page___________ of Question___________Student's # (PID)________________

Astro I:
Work out 3 out of 5 problems

Astro II:
Work out 3 out of 5 problems

(Partial credit will be given for partial answers)

My work is completed in full observance of the Honor code:

Signature________________________

Print name___________________________

CM-1

A point particle of mass $\boldsymbol{m}$ and charge $\boldsymbol{q}$ is attached to the end of a massless pendulum of length $\boldsymbol{l}$. The motion of the pendulum is confined to a plane. Let the pivot of the pendulum be fixed at a height $\boldsymbol{h}$ above an infinite horizontal conducting surface, with $\boldsymbol{h} > \boldsymbol{l}$. Ignore gravity in considering the motion of the pendulum.

1. Use an angular coordinate and obtain the Lagrangian.
2. Find the frequency of small amplitude motion.

CM-2

A particle of unit mass moves under the influence of gravity on the inner surface of the paraboloid of revolution $\boldsymbol{x^2 + y^2 = z}$, which is assumed to be frictionless. ($\boldsymbol{z}$ is the vertical direction.)

1. Obtain the equations of motion in cylindrical coordinates. (You do not have to solve them.)
2. What angular momentum must be given to the particle so that it describes a horizontal circle at the height $\boldsymbol{z = 1}$?

CM-3

Consider a satellite in circular Earth orbit and the stability of its orientation relative to the Earth. Let the mass of the Earth be $\boldsymbol{M}$ and the radius of the orbit be $\boldsymbol{r_0}$. By Kepler's third law the angular frequency of the orbit satisfies

$\boldsymbol{\Omega^2 = GM/r_0^3}$.

The satellite is an extended rigid body that can be idealized as two masses, $\boldsymbol{m}$, separated by a massless rigid rod of length $\boldsymbol{2a}$. The rod lies initially in the orbital plane and you should consider only motion in that plane. The satellite may rotate in the plane and therefore could corotate with its orbit, maintaining a fixed orientation with respect to the Earth.

1. Let the angle between the rigid rod and the direction to the Earth be given by $\boldsymbol{\Psi}$. Write down the kinetic energy of motion relative to the circular orbit of the center of mass (i.e., you are to take the center of mass motion as known).
2. Write down the potential energy of the rigid body and expand it in powers of $\boldsymbol{a/r_0}$ to find the leading non-vanishing $\boldsymbol{\Psi}$ dependent terms.
3. Write down the Lagrangian and obtain the equation of motion for satellite orientation (relative to the Earth; i.e., for motion in $\boldsymbol{\Psi}$).
4. Via an effective potential or other means find the equilibrium orientation angles. Which of these equilibria are stable and which are unstable?

CM-4

Consider a particle under the influence of a force that is constant in space but grows linearly with time. The corresponding Hamiltonian is

$$H = \frac{p^2}{2m} - \lambda x t\,,$$

where λ is a constant. Use the Hamilton-Jacobi method to find $q(t)$ for initial conditions $q(0) = q_0$ and $p(0) = p_0$.

Hint: You can separate variables if you add a term to the generating function to cancel the term containing xt. Try $S = S_x(x) + S_t(t) + \frac{1}{2}\lambda x t^2$ (where any dependence on the integration constant α is not shown). And you should be able to tell whether your answer makes sense.

CM-5

Determining motion by Taylor expansion and Poisson brackets:

1. Show for any function $A(q,p)$ and a time-independent Hamiltonian H that
$$\underbrace{[\,[\cdots[A,H],H],\cdots,H]}_{ntimes} = \frac{d^n A}{dt^n}\,.$$

2. Show that
$$q(t) = q(0)+[q,H]\Big|_{t=0} t+\frac{1}{2}[\,[q,H],H]\Big|_{t=0} t^2++\frac{1}{6}[\,[\,[q,H],H],H]\Big|_{t=0} t^3+\cdots\,,$$
as long as the series converges.

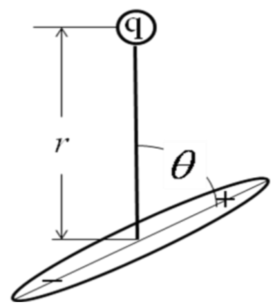
3. Without solving any differential equations, use the results from part (2) to obtain the solution $q(t)$ for a simple harmonic oscillator with spring constant k, mass m, and initial conditions $q(0) = q_0$, $p(0) = p_0$.

SM-1

The phonon modes of a crystal are treated as 3N independent harmonic oscillators. The associated energy is given by $E\{n_i\} = \sum_{i=1}^{3N}(n_i + 1/2)\hbar\omega_i$ and $n_i = 0,1,2\ldots$, and the distribution function of modes in angular frequency is given by $g(\omega)$ where $\int_0^\infty g(\omega)\,d\omega = 3N$. Show that the entropy associated with the phonons is given by $S = \frac{1}{T}\int_0^\infty \frac{\hbar\omega}{e^{\hbar\omega/kT}-1} g(\omega)\,d\omega - k\int_0^\infty \ln\left(1-e^{-\hbar\omega/kT}\right) g(\omega)\,d\omega$

SM-2

Compare two situations: (1) a charge q interacts with an electric dipole $\vec{p}$ that has a fixed orientation and is at a distance r, and (2) a charge q interacts with a dipole $\vec{p}$ that orients freely over all possible angles at a distance r. For case (1) the potential energy is given by $u(r)=\frac{qp\cos\theta}{4\pi\varepsilon_0}\frac{1}{r^2}$ where ε_0 is the permittivity and θ is shown below. Show that the interaction in case (2) is shorter-ranged than case (1) by showing that in case (2) $u(r)=-\frac{1}{3kT}\left(\frac{qp}{4\pi\varepsilon_0}\right)^2\frac{1}{r^4}$. Here, assume that |u(r)|<<kT, so that $e^{-u/kT}\approx 1-u/kT$.



SM-3

The partition functions of N particle (of mass m) classical ideal gas contained in a volume V and at temperature T is given by $Q_N(V,T)=\frac{1}{N!}\left(\frac{V}{\lambda^3}\right)^N$ where $\lambda=h/\sqrt{2\pi mkT}$. Show that the partition function of an ideal Fermi gas of two particles is $Q_N(V,T)=\frac{1}{2!}\left(\frac{V}{\lambda^3}\right)^2\left(1-\frac{1}{2^{3/2}}\frac{\lambda^3}{V}\right)$. Useful integral: $\int_0^\infty x^2e^{-\alpha x^2}dx=\frac{1}{2\alpha^{3/2}}\frac{1}{2}\sqrt{\pi}$.

SM-4

A particle of mass m with momentum (p_x,p_y,p_z) and coordinates (x,y,z) moves freely in a volume V. (a) Find the normalized distribution function $f(p_x)$ of the x-component of the momentum according to the classical micro-canonical ensemble with energy E.

(b) The corresponding canonical distribution (with temperature chosen to give averaged energy =E) is quite different from the micro-canonical distribution (you do NOT have to show this). Why do you think the two distributions are so different?

SM-5

A long vertical tube with a cross-section area $\boldsymbol{A}$ contains a mixture of $\boldsymbol{n}$ different ideal gases, each with the same number of particles **N**, but of different masses $\boldsymbol{m_k}$, $\boldsymbol{k=1,\ldots,n}$. Find a vertical position of the center of mass of this system in the presence of the Earth's gravity, assuming a constant altitude-independent free fall acceleration $\boldsymbol{g}$.

EMI-1

Consider a very long solenoid with radius *R*, *N* turns per unit length, and current *I*. Coaxial with the solenoid are two long cylindrical shells of length *l* - one, inside the solenoid at radius *a*, carries a charge +*Q* uniformly distributed over its surface; the other, outside the solenoid at radius *b*, carries charge –*Q*; *l* is supposed to be much greater than $b > R > a$. (Assume there is an electric field only in the region between the cylinders.) When the current in the solenoid is gradually reduced, the cylinders begin to rotate.

a) What are the torques on the outer cylinder and on the inner cylinder?

b) After the current is switched off, how much angular momentum have the two cylinders picked up?

c) Before the current is reduced, what is the total angular momentum in the fields?

EMI-2

A constant charge per unit length $\lambda = dQ/dz$ is distributed along an infinite-length insulator of negligible cross section. A charge ***q*** is present in the vicinity of the line charge.

1. Find the electric field and electric potential due to the line charge.
2. Determine how much work ***W*** is done on the point charge ***q*** if its distance from the line charge increases from cylindrical distance $R = a$ to distance $R = b$.
3. As $b \rightarrow \infty$, what happens to the work on the particle?
4. If the line charge were truncated to a finite total length of ***L*** (with ends at $z = \pm L/2$), give an approximate expression for the total work ***W*** done if the charge is taken radially away from $z=0$ and $R=a$ to $R=\infty$.

EMI-3

Calculate the interacting force between a dipole moment $\vec{p} = p\,\hat{z}$ and a conducting sphere of radius ***a***. The dipole moment is at a distance ***R (>a)*** away from the center of the sphere.

EMI-4

A charged sphere with radius *a* is placed in a media with dielectric constant ε. The charge distribution inside the sphere is given by $\rho(x, y, z) = \rho_0 \left(\frac{2z^2 - x^2 - y^2}{a^2}\right)$.

a) Show that the electric scalar potential along the *z*-axis outside the sphere is given by $V(x = 0, y = 0, z) = \frac{\rho_0}{35\varepsilon_0} \frac{a^5}{z^3}$.

b) Using the result in a) to find the general expression of the scalar potential $V(x, y, z)$ everywhere outside the sphere.

EMI-5

A spherical shell of permeability $\boldsymbol{\mu}$ is placed in a uniform field $\boldsymbol{B_o}$. If the internal and external radius of the shell are ***a*** and ***b***, respectively.

a) Find the magnetic field in the hollow interior. Be sure your solution reduces to the obvious result when ***a* =*b*** *(the shell is gone).*

b) Show that in the limit of large permeability the field is of order $\boldsymbol{B_o/\mu}$, thus this shell can act as a magnetic shield.

QM1-1

A spin-1/2 particle with magnetic moment $\boldsymbol{\mu}$ is in an eigenstate of $\boldsymbol{S_x}$ with eigenvalue ***h/2*** at time ***t = 0***. At that time it is placed in a magnetic field of magnitude ***B*** pointing in the z-direction and allowed to precess for time ***T***. At that time the magnetic field is rotated very, very rapidly, so that it now points in the y-direction. After another time interval ***T***, $\boldsymbol{S_x}$ is measured. What is the probability that it is found to be ***h/2***?

QMI-2

Four electrons are each localized to separate atoms in a crystal. The atoms are located at the corners of a regular tetrahedron, which is a triangular pyramid where each face is an equilateral triangle. The length of each edge is ***a***. Find the correction to the energy levels of the four electron system due to the spin-spin interaction between the electrons. You may assume that the spin-spin interaction term between any two electrons ***i*** and ***j*** is of the form:

$$H_{ij} = A\frac{\mathrm{S_i}\cdot\mathrm{S_j}}{r_{ij}^3}$$

where ***A*** is a constant and $\boldsymbol{r_{ij}}$ is the distance between the two electrons. You may also assume that ***a*** is much larger than the spatial extents of the electrons' wave functions, in other words the electrons are distinguishable by their atom's locations on the crystal lattice.

QMI-3

Consider a particle of mass ***m*** trapped in a one-dimensional simple harmonic oscillator well with a resonance angular frequency **ω**

Let |z> be a normalized eigenstate with eigenvalue ***z*** of the raising operator ***a***:

$$a = \frac{1}{\sqrt{2m}}(p + im\omega x)$$

Note that a is not a hermitian operator, hence z can be a complex number.
(a)Show that |z> satisfies the following relationship:

$$\sigma_x = m\omega\sigma_p$$

(b) Show that |z> satisfies the minimum uncertainty relationship between ***x*** and ***p***. You may find the result from part (a) useful for this part.
Comment: |z> is known as a coherent state and has many interesting properties.

QMI-4
Consider three distinguishable particles with spin 1/2 (and no spatial degrees of freedom).
(a) What are the possible values for the total angular momentum of all three particles? Are there any values that have more than one multiplet associated with them?
(b) Write explicit expressions for all the states in the basis that has definite values of the total angular momentum and z-projection.

QMI-5
Consider the one-dimensional Schroedinger equation with

$$V(x) = \begin{cases} \frac{m}{2}\omega^2 x^2 & \text{for} \quad x > 0, \\ +\infty & \text{for} \quad x < 0. \end{cases}$$

Find the energy eigenvalues.

EMII-1

Consider a pulse of electromagnetic radiation in vacuum. In a region within the midst of the pulse, the integrated momentum is

$$\vec{P} = \frac{1}{4\pi c} \int d^3x \, (\vec{E} \times \vec{B}),$$

and the integrated energy is

$$\mathcal{E} = \frac{1}{8\pi} \int d^3x \, (E^2 + B^2).$$

Assume that the following relationship holds between the integrated momentum and energy of the pulse:

$$|\vec{P}| \, c = \mathcal{E},$$

just like that of a single photon. From this relation (for the volume integrals) alone, show that $\vec{E} \cdot \vec{B} = 0$ and $E^2 = B^2$.

One of the crucial steps involves proving the inequality

$$(\vec{E} \times \vec{B})^2 \leq \frac{1}{4} (E^2 + B^2)^2.$$

EMII-2

Find the total cross section σ for the scattering of an electromagnetic wave of long reduced wavelength $\lambda/2\pi$ by a dielectric sphere of radius a (with $\lambda \gg a$) and permittivity ϵ.

You may recall that a dielectric sphere acquires a dipole moment $\vec{d}$ in a static electric field $\vec{E}$ given by

$$\vec{d} = \frac{\epsilon - 1}{\epsilon + 2} a^3 \, \vec{E}.$$

EMII-3

Two relativistic electrons with the same Lorentz factor γ approach each other obliquely. The particles have equal but opposite angles, $\pm\theta$, relative to the x axis. The electrons have just sufficient energy to create a π^+-π^- pair. After the collision the original two electrons emerge along with the newly created pair:

$$e^- + e^- \to e^- + e^- + \pi^- + \pi^+.$$

1. Determine how the Lorentz factor γ depends upon angle θ and the particle masses, m_e and m_π, if the reaction is just at threshold.

2. Determine the Lorentz factor γ' of the particles exiting the event.

3. Calculate γ for the two special cases: $\theta = 0$ and $\theta = \pi/2$.

EMII-4

The spectral-angular distribution of radiation from a relativistic electron is given by

$$\frac{dW}{d\omega d\Omega} = \frac{e^2\omega^2}{4\pi^2 c}\left|\int dt'\, \vec{n} \times (\vec{n} \times \vec{\beta}(t')) \exp[i\omega(t' - \vec{n}\cdot\vec{x}(t')/c)]\right|^2.$$

A perfect conductor fills the region $x > 0$. Empty space exists in the region $x < 0$. An observer is situated in empty space and confined to the x-y plane. The associated outward-directed unit vector is $\vec{n} = (\cos\theta, \sin\theta, 0)$. For such an observer, $\pi/2 < \theta < \pi$.

A relativistic electron $(-e)$, with velocity $\vec{\beta} = (\beta, 0, 0)$ in the vacuum region, approaches the conducting surface. The electron strikes the surface at the origin of coordinates at $t' = 0$ and abruptly disappears. This event causes transition radiation.

1. Since the electron was moving relativistically toward the conductor, and radiation will not propagate inside the conductor, explain why transition radiation can even exist in this problem.

2. Derive the spectral and angular distribution of the radiation that can be seen by the observer.

EMII-5

A circularly polarized light wave emerges from a laser and is propagating in vacuum. The light is nearly a plane wave directed along the z axis with wave number k. However, because the beam has finite transverse extent, the amplitude of the electric field drops off (gradually) in the x and y directions. The electric field can be approximately represented by

$$\vec{E} \simeq \frac{1}{\sqrt{2}} (\vec{e}_1 + i\vec{e}_2)\, E_0(x,y)\, e^{ikz}\, e^{-i\omega t} \;+\; a(x,y)\, \vec{e}_3\, e^{ikz}\, e^{-i\omega t},$$

where

$$\frac{1}{E_0} \left|\vec{\nabla} E_0(x,y)\right| \ll k.$$

The latter implies that the length scale for changes in the field in the x and y directions is many wavelengths $\lambda = 2\pi/k$ long.

1. Explain why the electric field must have a non-vanishing z component.
2. Show how the amplitude $a(x,y)$ depends upon $E_0(x,y)$.
3. Why is this only an approximate expression for the electric field?

QMII-1

A particle of mass m is trapped in a 2-dimensional infinite potential well with sides of length ***a***. The well is centered on the origin and the edges are parallel to the coordinate axes. The particle also experiences a "Gaussian wall" potential perturbation given as:

$$V'(x,y) = Ae^{-x^2/b^2}$$

where ***b*** << ***a*** and ***A*** is a constant with appropriate units. Use first order perturbation theory to find the ground and first excited state energy levels and their degeneracies. Approximate any integrals that you cannot evaluate easily. You may find the following integral useful:

$$\int_{-\infty}^{\infty} e^{-x^2/b^2}\, dx = b\sqrt{\pi}$$

QMII-2

In the interaction picture, the state $|\Psi(t)\rangle_I$ satisfies the equation i $d|\Psi(t)\rangle/dt = H_I(t)|\Psi(t)\rangle_I$

a) Derive an equation for the interaction picture evolution operator $U(t; t_0)$ where $|\Psi(t)\rangle_I = U(t; t_0)|\Psi(t_0)\rangle_I$ with $U(t; t) = 1$.

b) Solve the equation you have derived in (a) for $U(t; t_0)$ when the Hamiltonian $H_I(t)$

satisfies $[H_I(t_1), H_I(t_2)] \neq 0$ for $t_1 \neq t_2$. Define all symbols you use, and show why your solution is true.

QMII-3

Consider a charge particle with mass *m* and charge *q* in a one dimensional simple harmonic oscillator potential $V(x) = \frac{1}{2} m\omega x^2$. Initial the particle is in the ground state. Between $0 < t < \pi/\omega$ the particle is subject to a constant electric field E perturbation. Using the first order time dependent perturbation theory, calculate the probability of the particle in the eigenstate $|n>$ at the end of a perturbation period $T = \pi/\omega$.

QMII-4

(a) Under the parity operation $\boldsymbol{\pi}$ how the coordinate $\boldsymbol{x}$, momentum $\boldsymbol{p}$, and angular momentum $\boldsymbol{L}$ transform?

(b)The ground state of a SHO $|0>$ is known to be parity even. Show that, in general, the excited state $|n>$ is parity even/odd, depending on whether n is an even or odd integer.

(c) Under time reversal operation $\boldsymbol{\theta}$, how the coordinate $\boldsymbol{x}$, momentum $\boldsymbol{p}$, and angular momentum $\boldsymbol{L}$ transform?

(d) A spin half particle is in a state $|\alpha >= a|+> +b|->$. What condition does one need to impose on the complex numbers a, b if the state is invariant under time reversal symmetry?

QMII-5

An isolated hydrogen atom has a hyperfine interaction between the proton and electron spins ($\mathbf{S}_1$ and $\mathbf{S}_2$, respectively) of the form $J\mathbf{S}_1.\mathbf{S}_2$. The two spins have magnetic moments $\alpha\, \mathbf{S_1}$ and $\beta\, \mathbf{S_2}$, and the system is in a uniform magnetic field $\boldsymbol{B}$. Consider only the orbital ground state.

(a) Find the exact energy eigenvalues of this system and sketch the hyperfine splitting spectrum as a function of magnetic field.

(b) Calculate the eigenstates associated with each level.

AstroI-1

Basic Astronomy

1. A new planet is discovered! Planet X is observed to orbit the sun every 300 years. What is the semi-major axis of Planet X's orbit in AU?

2. Planet X is in a circular orbit. Given that 1 AU = 1.5 × 1011 m, what is the distance to Planet X in meters when at closest approach to Earth?

3. When at closest approach to Earth, Planet X is observed to be 3.8 arcseconds in diameter. What is Planet X's diameter in meters?

4. What is the diameter of Planet X in Earth diameters and in AU? (The diameter of Earth is 1.3 × 107 m.)

5. Planet X is observed to have a small moon. This moon is observed to orbit the planet once per month at a distance of 15 Planet X diameters. What is the mass of Planet X in solar masses? (Assume that the mass of the moon is negligible in comparison.)

6. What is the mass of Planet X in kilograms and in Earth masses? (The mass of the sun is 2.0 × 1030 kg. The mass of Earth is 6.0 × 1024 kg.)

7. What is the average density of Planet X in kg/m3?

8. Based only on your calculated values for the diameter, mass, and density of Planet X, it is probably a:

A. Large comet
B. Large asteroid
C. Kuiper belt object similar to Pluto
D. Terrestrial planet similar to Earth
E. Jovian planet similar to Saturn
F. Jovian planet similar to Jupiter
G. Jovian planet similar to a giant Jupiter
H. Small star

AstroI-2

Consider a model star in which the density is a linear function of radius:
$\rho(r) = \rho_c[1 - r/R]$, where ρ_c is the central density and R is the total stellar radius at which P(R) = T(R) = 0.

a. Find an expression for the central density in terms of total radius R and total mass M

b. Use the equation of hydrostatic equilibrium and zero boundary conditions to find pressure as a function of radius. Write an expression for the central pressure in terms of R and M
c. What is the central temperature (assume ideal monoatomic gas equation of state).
d. Verify that this linear density model obeys the corollary to the virial theorem: $U = -\Omega/2$ where U is the total internal energy and Ω is the gravitational potential energy.

AstroI-3
Equilibrium of White Dwarfs

a. Derive the equation of hydrostatic equilibrium

b. Use dimensional arguments and the result of part a to derive the mass-radius relationship for a fully (non-relativistic) degenerate white dwarf

c. What is the slope of a white dwarf cooling track in the Log L, Teff plane (H-R diagram)? Make a plot of a white dwarf cooling track in this diagram. For this exercise you may assume a 0.6 solar mass white dwarf passes through solar temperature with a radius of 0.01 Rsun.

d. Now plot cooling tracks for white dwarfs of 1.1, 0.8 and 0.4 solar masses. Write the expression for the dependence of Log L on M for constant T.

AstroI-4
Virial Theorem
There is a commonly-cited corollary to the virial theorem as applied to spherical stars in hydrostatic equilibrium with ideal mono-atomic gas equations-of-state.

a. Re-derive this theorem under the following assumptions:

Ideal diatomic gas
Completely relativistic gas

b. The corollary to the Virial Theorem is often used to claim that a star powered by gravitational contraction will use ½ of the gravitational energy released to heat up, and the other half is radiated away. Explain why this is true for the ideal mono-atomi gas, and modify the statement for the two cases you have derived.

c. Physically speaking, why can there be no simple expression or statement like this for non-relativistic electron degenerate stars (i.e. white dwarfs).

AstroI-5
Lifetimes of Stars

a. Assuming the mass-luminosity relationship on the main sequence is $L \propto M^{3.5}$, derive a relationship for lifetime on the main sequence in terms of total M under the assumption that all stars have the same fraction of their H mass available for nuclear burning

b. Derive a similar relationship for stars powered by gravitational contraction alone (your expression will contain total mass and total radius).

c. Draw an H-R diagram that shows rough isochrones for co-eval populations 1, 5 and 10 Gyr after birth.

d. Now suppose there exist clusters made of pure iron stars with masses like those of normal stars. Describe the evolution of these stars as they follow Kelvin-Helmholtz contraction, and draw some isochrones for different age populations (please give the age in years for the isochrones you show, which obviously requires that you calculate the Kelvin-Helmholtz timescale with real numbers)

AstroII-1
Relaxation in a Galaxy Cluster

Consider an idealized galaxy cluster in which all galaxies have the same mass (10^{12} M_{sun}). The cluster contains 2000 galaxies within a radius of 2Mpc and has line-of-sight velocity dispersion 1000 km/s (somewhat like the Virgo Cluster).

(a) Draw a diagram of a two-galaxy encounter in the weak encounter limit (impulse approximation), in the rest frame of one of the galaxies. Call their relative velocity v. Show the impact parameter b, and use a long arrow to indicate the trajectory of the non-rest-frame galaxy.

(b) What is the change in velocity for each galaxy (make sure to answer for both)? Distinguish the parallel and perpendicular components. Note that $\int dx/(c_1+c_2x^2)^{3/2}$ $=2/(c1\sqrt{c2})$ for integration limits of $x=-\infty$ to $+\infty$.

(c) The weak encounter approximation breaks down if $\Delta v \sim v$, the typical 3D velocity of a galaxy in the cluster. What impact parameter does this breakdown occur at? Compare this “strong encounter” impact parameter to the typical distance between galaxies.

(d) Approximately calculate the two-body relaxation time for this cluster. Comparing this number to the age of the Universe, comment on how the cluster has achieved a relaxed, roughly spherical configuration.

AstroII-2
Closed Box Star Formation

(a) Demonstrate that an exponential star formation history results from the assumption that the star formation rate is proportional to the gas mass in a "closed box" system, i.e. that $\frac{dM_s}{dt}(t) = kM_g(t)$.
(b) Assume the metallicity Z increases as the gas is consumed with yield p, such that $Z(t) = -p\ln\frac{M_g(t)}{M_g(0)}$. Derive the time dependence of Z in terms of p and k. Compute the yield p for which the enrichment timescale for the metallicity to reach the solar value Z=0.02 is the same as the gas depletion timescale over which the gas drops by 1/*e*.
(c) If we allow for external gas infall (open box model), in which direction will the yield change from the value you computed in part b?

AstroII-3
A Young Massive Star Cluster
A massive ($7\text{x}10^7 M_{sun}$) star cluster with a light-weighted simple stellar population age of ~500 Myr is 6 kpc away on-sky from an elliptical galaxy with multiple tidal streams and shells, at the same redshift. The galaxy has dispersion ~135 km/s, equivalent to rotation velocity ~200 km/s.
(a) In the Chandrasekhar approximation, the drag force on this star cluster from dynamical friction with particles in a dark matter halo is $F_{df} = -\frac{4\pi G^2 M_{sat}^{\ 2}}{V_{sat}^{\ 2}}\rho\ln\Lambda$, where ρ is the mass density of dark matter particles and $\ln\Lambda \sim 3$. Prove that if the cluster starts on a nearly stable circular orbit, the time for the cluster to sink to the center of the galaxy is $t_{\sin k} = \frac{r^2 V}{18GM_{sat}}$, where V is the galaxy rotation velocity (which may be assumed constant at all radii).
(b) If the cluster stellar population is indeed "simple," is the light-weighted age surprising compared to the value of the sinking time, evaluated from the equation in part a? What if the stellar population turns out to be composite? Comment on how your answers are affected by the fact that the 6kpc distance is actually a lower limit due to projection on the sky.
(c) The cluster is virialized and its light profile resembles a cE like M32. Will it follow the Faber-Jackson relation of elliptical galaxies, or if not, how will it deviate? In your answer, explicitly discuss each assumption required to derived the Faber-Jackson relation.

AstroII-4
Spherical Gas Clouds
(a) A pressureless, uniform-density spherical gas cloud collapses under gravity in the free-fall timescale t_{ff}. Show that $t_{ff} = \sqrt{3\pi/(32G\rho)}$ using Kepler's 3rd Law $P^2 \propto a^3$,

where ρ is the density of the cloud, P is the period of an orbit around a point mass, and a is the semi-major axis of an orbit around a point mass.
(b) Now suppose the cloud is not pressureless, but is supported by internal random motions with typical dispersion equal to the sound speed v. Write down an order-of-magnitude inequality describing the range of cloud sizes that remain unstable to collapse.
(c) A borderline stable molecular cloud has density ρ_A, size $l_A=2r_A$, and internal sound speed v_A. If cloud B is 6x smaller and 16x denser, prove that its sound speed v_B must be $0.67v_A$ to achieve the same borderline stability.
(d) If the speed of sound v_s is related to temperature T in a molecular gas cloud by $v=\sqrt{1.4k_BT/m}$ where m is the mass of a typical molecule and k_B is Boltzmann's constant, how do the wavelengths of peak blackbody emission for the two clouds in part c compare? What properties of the clouds suggest assuming blackbody emission is reasonable?

AstroII-5
Vertical Motion in a Spiral Galaxy Disk
A gas cloud plunges through a spiral disk that has scale height h_z = 350pc. The interaction creates young star clusters in the spiral disk, extending above it by ~100pc. The cloud emerges at z=500 pc.
Assume that at small heights z above the disk, the spiral disk potential takes the approximate form $\varphi = 4\pi G\rho_0 h_z^2 e^{-\frac{|z|}{h_z}} + 4\pi G\rho_0 h_z |z|$. For a realistic mass density ρ_0 ~ 0.1M_{sun}/pc^3 this means $\sqrt{\varphi(z=0)} = \sqrt{4\pi G\rho_0 h_z^2} \sim 25$ km/s.

(a) Expand the potential in a Taylor series around z=0 to show that the force equation is $F = -4\pi G\rho_0 z\left(1-\frac{z}{2h_z}\right)$.
(b) Will the newly formed star clusters experience simple harmonic motion? What is the third integral that is conserved for their orbits? What about the gas cloud?
(c) Use the Poisson equation appropriate for a thin-disk system to determine ρ.
(d) Show that the surface mass density is $2h_z\rho_0$.

Useful facts:
$F = Gm_1m_2/d^2$
G=4.3 x 10^{-6} kpc $(km/s)^2/M_{sun}$
1kpc = 1Gyr x 1km/s
Virial Theorem 2KE+PE=0
Poisson Equation $\nabla^2\varphi = -4\pi G\rho$
Wien's Law T = 3 mm·K / λ_{peak}
centripetal force $F=mV^2/r$ (uniform circular motion)
Faber-Jackson Relation $L \propto \sigma^4$
Dynamical time $t_{dyn}=\sqrt{2}t_{ff}$
Crossing time t_{cross} = R/V = 1 Gyr (**R** in kpc/**V** in km/s)
Relaxation time t_{relax} = 0.1N / ln(N) t_{cross} = 10^6 yr x 0.1N/ln(N) x (**R** in pc/**V** in km/s)